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The combined effect of J_2 and C_{22} on the critical inclination of a lunar orbiter

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Abstract

The lunar orbiter exhibits peculiar dynamical features: on one hand, the C_{22} lunar gravity term has to be considered – since it is only 1/10 of the J_2 term – and on the other hand the third body effect of the Earth is quite large.

We use the Lie Transform as perturbation method; the solution is developed in powers of the small factors linked to J_2 and C_{22}. Series development are made with our home-made Algebraic Manipulator, the MM (standing for "Moon's series Manipulator"). The results are obtained in a closed form, without any series developments in eccentricity or inclination. So the solution apply for a wide range of values, except for few isolated critical values.

In this paper, we focus on the first order results in J_2 and C_{22}. It has been found that the critical inclination $I_c = 63°26'$ of the zonal problem may be strongly affected by the C_{22} coefficient and by the value of the longitude of the ascending node h; I_c^* may be in the whole range from 0° to 90°, depending on the value of the ratio $\sigma = J_2/C_{22}$.

This result may be used for a wide range of primaries within the frame of the artificial satellite theory, including natural satellites in synchronous rotation, at hydrostatic equilibrium ($\sigma = 10/3$) or not – like the Moon ($\sigma = 9.07$). It may even be applied to Asteroids, like the famous case of Asteroid 433 Eros ($\sigma = 2.20$), around which the NEAR-Shoemaker spacecraft flew recently.

Second order results in J_2 and C_{22} have already been described elsewhere [De Saedeleer, B. Analytical theory of an artificial satellite of the moon, in: Belbruno, E., Gurfil, P. (Eds.), Astrodynamics, Space Missions, and Chaos – Volume 1017 of the Annals of the New York Academy of Sciences – Proceedings of the Conference New Trends in Astrodynamics and Applications, January 20–22, 2003, Washington, pp. 434–449, 2004], and the generator $\mathcal{W}_2$ in ϵ^2 has also been validated by the results of Kozai [Kozai, Y. Second-order solution of artificial satellite theory without air drag. Astron. J. 67, 446–461, 1962].

Keywords: Lunar artificial satellite; Lie; Hamiltonian; Non-zonal; C_{22}; Critical inclination

1. Introduction

The case study of a satellite around the Moon is quite different than the one around the Earth on several aspects. First of all, the moon is a slowly rotating body and has no dense atmosphere. Secondly, as it is well known, the lunar gravity field is far from being central, nor exhibits any strong symmetry of revolution; see, e.g., Konopliv et al. (2001) for a recent model in spherical harmonics. The order of magnitude of the second order coefficients for the Earth (Kaula, 1966, p. 115) and the Moon (Bills and Ferrari, 1980, p. 1018) is given in the Table 1. The Moon is much less flattened than the Earth, which makes the C_{22} coefficient to come closer to J_2 (at 1 order of magnitude instead of 3 in the case of the Earth); so it needs to be considered. Moreover, the effect of the Earth on the lunar satellite is quite large.

Our work has been conducted in the frame of the *General Perturbations*; the analytical methods still being of great importance (Wnuk, 1999) for several reasons,

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Table 1
Some orders of magnitude for J_2 and C_{22}; SSHE means Synchronous Satellite in Hydrostatic Equilibrium

	$C_{20} \equiv -J_2$	C_{22}	$\sigma = J_2/C_{22}$
Earth ⊕	-1083×10^{-6}	1.57×10^{-6}	689.81
Moon ☾	-202×10^{-6}	22.26×10^{-6}	9.07
SSHE	$-J_2$	C_{22}	10/3
433 Eros	−0.117344	0.053278	2.20

among which: (i) the functional character of various perturbations effects may be examined only with the use of analytical formulas for perturbations, and (ii) numerical integrations done in the frame of the *Special Perturbations* may quickly become costly in terms of computer time, in spite of the unprecedented increase in computational power. The analytical methods are sometimes less accurate but they give a more general picture of the set of orbits one could examine after by more accurate numerical investigations; numerical and analytical methods are therefore complementary.

We used the now common analytical method of the Lie transform; while others used the method of von Zeipel (Marchal, 2000). Semi-analytical methods have also been used in order to average the potential (D'Avanzo et al., 1997); while others proceed to integration of some variational equations of the keplerian elements (Liu and Wang, 2000). There is also a wide use of Lagrange planetary equations (Meyer et al., 1994), but we prefer the canonical frame of the Lie transform .

As we want analytic solutions, the model Hamiltonian has to be as simple as possible, yet capable to account accurately for the observed orbital behavior of real satellites. This question could be answered in a next step, by performing a numerical integration of the averaged equations and comparing them to published ephemeris.

The long term effects on a lunar satellite have already been documented at some level in the literature; it is of contemporary interest since current exploration goals plan a permanent lunar outpost requiring longer lifetime of lunar satellites, like putting a lunar transfer vehicle into a parking orbit during the lunar stay (typically 6 months). It is of crucial interest to know the long term evolution of that parking orbit in order to be able to correctly plan rendezvous maneuvers with lunar landers. This long term situation is contrast with the short stay times (1–3 days) of the Apollo missions.

The search for frozen orbits is a subject of inalterable interest since these orbit allow to minimize the cost of the orbit maintenance of the spacecraft (Coffey et al., 1994). The concept of a "frozen orbit" itself needs sometimes to be specified, depending on which quantity is considered to be close to stationary: either the eccentricity or the argument of periapsis, or even both (Elipe and Lara, 2003). Let's quote that by definition, a frozen orbit makes reference to an averaging procedure. Several authors make numerical continuation of frozen orbits (Elipe and Lara, 2003; Lara et al., 1995), giving access to a wide range of frozen orbits at any inclination.

The worse consequence of a perturbation would be to increase the eccentricity in such a way that the altitude may rapidly fall down up to the end of life of the satellite by collision onto the primary, in several months (Liu and Wang, 2000), or even in several days (Meyer et al., 1994). The increase in eccentricity may be due either to the effect of higher zonal harmonics on a low satellite (D'Avanzo et al., 1997), especially J_5 (Meyer et al., 1994) or to the big third body effect on a high satellite (Liu and Wang, 2000).

The third body perturbation will not be considered in the frame of this article (but well in a forthcoming one), although it is known that it may affect the dynamics of high altitude satellites, e.g., the stability of the frozen orbits, for example by changing the argument of perilune (D'Avanzo et al., 1997).

Another interesting point of view is that a higher oblateness may extend the lifetime of a satellite as it counteracts the third body effect (Liu and Wang, 2000; Marchal, 2000); this may happen at different altitude, let's say of the order of 2000 km.

The initial conditions (inclination and argument of perilune) should also be carefully considered, as they play an important role on the orbital lifetime. It has been shown that polar orbits have shorter lifetimes (Liu and Wang, 2000; Marchal, 2000) and that some range of inclinations may be more favorable than others (Meyer et al., 1994).

In order to achieve a suitable analytical theory, we considered here only the two main greatest spherical harmonics: J_2 and C_{22}. Although the introduction of higher order harmonics could always improve the accuracy, they were not included for several reasons:

(i) They should introduce many additional perturbation variables, more heavy to handle, which exponentially increase the size of the computations, while it does not give necessarily additional relevant analytical information.
(ii) Their order of magnitude is anyway lower: for example: $J_5/J_2 < (|J_4/J_2| \approx J_3/J_2) < 5 \times 10^{-2}$; while the next biggest harmonic to be considered after J_2 and C_{22} should be C_{31} since $C_{31}/J_2 \approx 0.14$.
(iii) Their exact value still need to be more accurately determined; let's note that all existing lunar gravity models suffer anyway limitations due to a lack of satellite tracking data from the far side of the moon (Meyer et al., 1994).
(iv) They could always be progressively quite easily incorporated afterwards if required.

More generally, one should be aware that there is a trade-off to find between the analytical feature and the level of accuracy.

So in this paper, we will deliberately focus on the first order results in the perturbations J_2 and C_{22}, while the second order results have already been described elsewhere (see De Saedeleer, 2004) to some extend.

The following assumptions have been made: the orbit of the Moon is circular; the motion of the Moon is uniform; the lunar equator lies in the ecliptic; the perturbation of the Sun is negligible; and the longitude of the lunar longest meridian λ_{22} is equal to the longitude of the Earth $\lambda_\oplus$ (librations are neglected). This last assumption is one of the well known Cassini's laws (Cook, 1988), stating that the Moon is in synchronous rotation.

This last feature imposes a given order of magnitude for the ratio $\sigma = J_2/C_{22}$: this ratio is exactly 10/3 when the satellite is assumed to be in hydrostatic equilibrium.[1] This is a common assumption, which has also widely been used for Jovian satellites (Schubert et al., 1994); in particular see (Anderson et al., 1998) for Europa and (Anderson et al., 1997) for Callisto. Note that the Moon is clearly non-hydrostatic (Anderson et al., 2003; Hubbard and Anderson, 1978), with a value of $\sigma = 9.07$ instead of 10/3. The result could also be applied to Asteroids, like the famous case of Asteroid 433 Eros $(\sigma = J_2/C_{22} = -2\sqrt{3} \times \overline{C}_{20}/\overline{C}_{22} = 2.20)$, around which the NEAR-Shoemaker spacecraft flew recently (Miller et al., 2002).

2. Partial perturbative Hamiltonians

We work within the frame of the Hamiltonian formalism and use the classical Delaunay canonical variables $(q_i, p_i) = (l, g, h, L, G, H)$. The term $\frac{1}{2}v^2 - \frac{\mu}{r}$ is then simply written $\mathcal{H}_0^{(0)} = -\mu^2/2L^2$ (the unperturbed potential); next one has to develop the perturbations.

We define an inertial frame (x, y, z) as follows (see Fig. 1): the origin is taken at the center of the Moon; the x direction is the one of the first point of Aries ϒ, the y is the direction normal to x and contained in the lunar equatorial plane containing x and the z direction is the right-handed normal to (x, y). We define also the spherical coordinates (r, λ', ϕ). The zonal perturbation in J_2 is defined as usual by $\epsilon\mathcal{H}_2^{(0)} = \epsilon(\mu/r^3)P_{20}(\sin\phi)$ where we use $\epsilon = J_2R^2$ and the Legendre Associated Functions P_{nm}. The argument $(\sin\phi)$ may partially be translated into Delaunay variables by way of spherical trigonometry (see Fig. 1, where the plane of the orbit is at an inclination I): $\sin\phi = \sin I \sin(f + g)$.

The sectorial perturbation in C_{22} requires us to define λ_{22} as the longitude of the lunar longest meridian (minimum inertia), which is $\lambda_{22} = \lambda_\oplus$. Since this angle rotates at the rate of the synchronous rotation $\dot{\lambda}_\oplus = n_☾$, we preferably introduce a rotating frame by defining $\lambda = \lambda' - \lambda_\oplus$ and also $h = \Omega - \lambda_\oplus$. A new term must then be added to the Hamiltonian in order to have $\dot{h} = \partial\mathcal{H}/\partial H = -n_☾$. Now the sectorial perturbation may be written as $\delta\mathcal{HB}_2^{(0)} = \delta\mu r^{-3}P_{22}(\sin\phi)\cos(2\lambda)$, where we define $\delta = -C_{22}R^2$. We then use again some spherical trigonometry to switch into the canonical variables and we introduce the useful shortcuts $(s, c) = (\sin I, \cos I)$. In summary, we will write in our case:

$$\mathcal{H}^{(0)} = \mathcal{H}_0^{(0)} + \epsilon\mathcal{H}_1^{(0)} + \delta\mathcal{HB}_1^{(0)} - n_☾H \tag{1}$$

along with the definitions:

$$\mathcal{H}_1^{(0)} = \frac{\mu}{4r^3}(1 - 3c^2 - 3s^2\cos(2f + 2g)), \tag{2}$$

$$\mathcal{HB}_1^{(0)} = \frac{3\mu}{4r^3}\{2s^2\cos(2h) + (c+1)^2\cos(2f + 2g + 2h) + (c-1)^2\cos(2f + 2g + 2h)\} \tag{3}$$

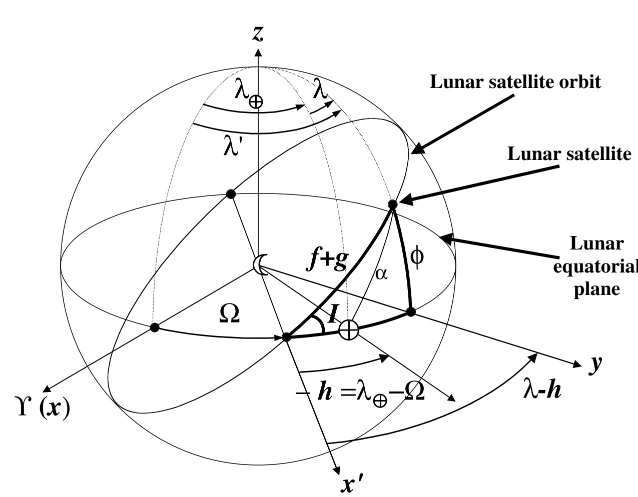

Fig. 1. Simplified selenocentric sphere.

[1] The theory of equilibrium figures (it is assumed that the source of the J_2 and C_{22} is an equilibrium ellipsoidal distorsion due to spin and tidal forces with synchronous rotation; both contributions being added linearly) for synchronous rotating satellites is well-known (Zharkov et al., 1985) and yields the following relationships: $J_2 = (5/6)\alpha K_2$ and $C_{22} = (1/4)\alpha k_2$, hence $J_2 = (10/3)C_{22}$, where α is a quantity which depends on the internal density distribution ($\alpha = 1/2$ for a constant density), and k_2 is the Love number describing the rheology. Similar relationships do exist to describe the triaxial shape of the satellite (Hubbard and Anderson, 1978), and in particular the semi-major axes of the ellipsoid (Zharkov et al., 1985).
Conclusions could then be drawn on the differentiation of the satellite when analysing the departure from 0.4 (homogenous body) of C/MR^2 computed as a function of α (the approximate Radau relationship, see Anderson et al. (2001)) knowing the rotational parameter $q_r = \omega^2R^3/GM$. The validity of the assumption of hydrostatic equilibrium may be discussed. It is always possible indeed that some satellites deforms non-hydrostatically to tidal and rotational forcing (like when the satellite is capable of maintaining elastic stresses over geologic times), or have a non-homogenous internal structure like a fossil tidal bulge.These non-hydrostatic effects (Mueller and McKinnon, 1988) could give rise to inconsistency within the hydrostatic theory. For example, a feasible diagnostic to test the validity of the assumption is to take two independent sufficiently accurate measurements of α, by measuring J_2 and C_{22} separately, using, respectively, a polar and an equatorial orbit. If the two values of α differ too much, or are unrealistic, then the hydrostatic equilibrium assumption should be rejected (Hubbard and Anderson, 1978).

Table 2
Table of partial derivatives

	ξ	a	n	s	c	e	η	f
$\partial/\partial L$	$\frac{\xi^2\eta^2}{na^2e}\cos f$	$\frac{2}{an}$	$-\frac{3}{a^2}$	0	0	$\frac{\eta^2}{na^2e}$	$-\frac{\eta}{na^2}$	$\frac{1+\xi\eta^2}{na^2e}\sin f$
$\partial/\partial G$	$-\frac{\xi^2\eta}{na^2e}\cos f$	0	0	$\frac{c^2}{na^2\eta s}$	$-\frac{c}{na^2\eta}$	$-\frac{\eta}{na^2e}$	$\frac{1}{na^2}$	$-\frac{1+\xi\eta^2}{\eta na^2e}\sin f$
$\partial/\partial H$	0	0	0	$-\frac{c}{na^2\eta s}$	$\frac{1}{na^2\eta}$	0	0	0
$\partial/\partial l$	$-\frac{\xi^2 e}{\eta}\sin f$	0	0	0	0	0	0	$\xi^2\eta$

note that we do not consider $n\mathscr{C}_H$ as a perturbation in this article.

We come back on the choice of the variables now. There remains the variable r and f to be expressed as a function of (l,g,h) in order to be able to apply a canonical perturbation method. It turns out that the functions $r=r(l,g,h)$ and $f=f(l,g,h)$ cannot be expressed in a closed form; so we prefer to use the following set of auxiliary variables $(\xi,f,g,h,a,n,e,\eta,s,c)$, which is closed and allow high eccentricities:

$$\begin{array}{ll} \xi=\frac{a}{r}=\frac{1+e\cos f}{1-e^2}=\frac{1}{1-e\cos E} & f \\ a=\frac{L^2}{\mu} & n=\frac{\mu^2}{L^3} \\ e=\sqrt{1-\left(\frac{G}{L}\right)^2} & \eta=\sqrt{1-e^2}=\frac{G}{L} \\ s=\sin I=\sqrt{1-\left(\frac{H}{G}\right)^2} & c=\cos I=\frac{H}{G} \\ g & h \end{array} \tag{4}$$

The only drawback of this set (4) is that it is redundant and that we need to perform partial derivatives of them with respect to the canonical variables (l,g,h,L,G,H); but it is not too burdensome; the result is given in the Table 2. Note that the quantity $\frac{\partial f}{\partial l}=\xi^2\eta$ plays an important role, since it will allow us to switch the integration from l to f. In this new set of variables (4), the factor μr^{-3} appearing in (2) and (3) may be written $\xi^3 n^2$.

3. Perturbation method using several parameters

We use here the Lie Transform (Deprit, 1969) as canonical perturbation method, with the parameters ϵ and δ. The initial Hamiltonian (*input*) is written $\mathscr{H}^{(0)}=\sum_{i\geqslant 0}\frac{\epsilon^i}{i!}\mathscr{H}_i^{(0)}$; while the transformed Hamiltonian (*output*) is written $\mathscr{H}_0=\sum_{i\geqslant 0}\frac{\epsilon^i}{i!}\mathscr{H}_0^{(i)}$. This transformation is symbolized by the Lie triangle (see Fig. 2).

The triangle is filled by the way of the classical recursive formula, where the $\mathscr{W}_k$ are the generating functions and $(A;B)$ is the Poisson parenthesis. We may write $\mathscr{H}_0^{(i)}$ as $\overline{\mathscr{H}}_0^{(i)}$ in order to remember that the fast angle l has been eliminated; we always put the periodic part in the generator $\mathscr{W}_i$.

$$\begin{array}{lllll} \mathcal{H}_0^{(0)}=-\frac{\mu^2}{2L^2} & & & & \\ \epsilon\mathcal{H}_1^{(0)}+\delta\mathcal{HB}_1^{(0)} & \mathcal{H}_0^{(1)} & & & \\ 0 & \mathcal{H}_1^{(1)} & \mathcal{H}_0^{(2)} & & \\ 0 & \mathcal{H}_2^{(1)} & \mathcal{H}_1^{(2)} & \mathcal{H}_0^{(3)} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Fig. 2. The Lie triangle.

For the first order, we have:

$$\epsilon\mathscr{H}_0^{(1)}+\delta\mathscr{HB}_0^{(1)}=\epsilon\mathscr{H}_1^{(0)}+\delta\mathscr{HB}_1^{(0)}+\left(\mathscr{H}_0^{(0)};\mathscr{W}_1\right) \tag{5}$$

and we choose:

$$\epsilon\overline{\mathscr{H}}_0^{(1)}+\delta\overline{\mathscr{HB}}_0^{(1)}=\frac{1}{2\pi}\int_0^{2\pi}\left(\epsilon\mathscr{H}_1^{(0)}+\delta\mathscr{HB}_1^{(0)}\right)\mathrm{d}l=\ldots \tag{6}$$

while $(\mathscr{H}_0^{(0)};\mathscr{W}_1)$ reduces to $n\frac{\partial\mathscr{W}_1}{\partial l}$ which has then to be integrated with respect to l.

Higher orders may be achieved by the same way, provided we are able to compute the integrals. Some tricks are given in the literature, like (Aksnes, 1971). The terms containing the factor $(f-l)$ receive a special treatment, as $(f-l)$ is indeed well known to play an important role in the problem of the artificial satellite (see Metris, 1991). The second order will contain the combinations of perturbation parameters $(\epsilon^2,\delta^2,\epsilon\delta)$.

4. Results obtained by the symbolic manipulation software MM

We used a specific FORTRAN code called the MM, standing for "Moon's series Manipulator", which has been developed at our University. In this tool, each expression is given by a series of linear trigonometric functions, with polynomial coefficients. The concern to keep linear expression is motivated by the fact that we want

Table 3
The initial perturbation $\epsilon\mathscr{H}_1^{(0)} + \delta\mathscr{HB}_1^{(0)}$

	f	g	h	ξ	a	n	e	η	c	s	$(f-l)$	n_δ	Coefficient
cos	0	0	0	3	0	2	0	0	0	0	0	0	0.25×10^0
cos	0	0	0	3	0	2	0	0	2	0	0	0	-0.75×10^0
cos	2	2	0	3	0	2	0	0	0	2	0	0	-0.75×10^0
cos	0	0	2	3	0	2	0	0	0	0	0	1	0.15×10^1
cos	0	0	2	3	0	2	0	0	2	0	0	1	-0.15×10^1
cos	2	2	2	3	0	2	0	0	0	0	0	1	0.75×10^0
cos	2	2	2	3	0	2	0	0	1	0	0	1	0.15×10^1
cos	2	2	2	3	0	2	0	0	2	0	0	1	0.75×10^0
cos	2	2	−2	3	0	2	0	0	0	0	0	1	0.75×10^0
cos	2	2	−2	3	0	2	0	0	1	0	0	1	-0.15×10^1
cos	2	2	−2	3	0	2	0	0	2	0	0	1	0.75×10^0

Table 4
The first order averaged Hamiltonian $\epsilon\overline{\mathscr{H}}_0^{(1)} + \delta\overline{\mathscr{HB}}_0^{(1)}$

	f	g	h	ξ	a	n	e	η	c	s	$(f-l)$	n_δ	Coefficient
cos	0	0	0	0	0	2	0	−3	0	0	0	0	0.25×10^0
cos	0	0	0	0	0	2	0	−3	2	0	0	0	-0.75×10^0
cos	0	0	2	0	0	2	0	−3	ϕ	0	0	1	0.15×10^1
cos	0	0	2	0	0	2	0	−3	2	0	0	1	-0.15×10^1

to keep easy integrations. An example of such a series is given in Table 3. It is of course impossible to give all the results here, since the series may contain a lot of terms, but we give explicitly some of them here, with a focus on the first order, while we give references for the others.

4.1. First order results: in J_2 and C_{22}

The series of the original perturbation in the parameters J_2 and C_{22} written in (2) and (3) is given in Table 3.

Note that the variable n_δ represent the exponent in the perturbative parameter δ. For an expression of order p, we then have terms containing factors like $\epsilon^{p-n_\delta}\delta^{n_\delta}$. Note also that we keep as long as possible the variable ξ as such in order to have a compact form for the series. The result of the computation (6) of the first order averaged Hamiltonian is given in Table 4, which can be rewritten in a more usual form:

$$\epsilon\overline{\mathscr{H}}_0^{(1)} + \delta\overline{\mathscr{HB}}_0^{(1)} = \epsilon\frac{n^2}{4\eta^3}(1-3c^2) + \delta\frac{3n^2}{4\eta^3}(2s^2\cos(2h)). \tag{7}$$

This form was expected: looking at (2) and (3) with $\mu r^{-3} = \xi^3 n^2$ and considering that the constant part of ξ^3, after being divided by $\xi^2\eta$ gives finally η^{-3}.

Of course, in the case $\delta = 0$ we retrieve the known results for the effect of J_2 only. The expression of $\epsilon\overline{\mathscr{H}}_0^{(1)}$ is the same[2] as the one given by $(-F_1^*)$ defined in (13) of Brouwer (1959), when setting $2k_2 = \epsilon$. It is also consistent with the more general formula valid for any zonal harmonic (see De Saedeleer, 2005).

Moreover, when expressing the averaged Hamiltonian as

$$\overline{\mathscr{H}} = \overline{\mathscr{H}}_0^{(0)}(\overline{L}) + \epsilon\,\overline{\mathscr{H}}_0^{(1)}(\overline{L},\overline{G},\overline{H}) + \delta\,\overline{\mathscr{HB}}_0^{(1)}(\overline{L},\overline{G},\overline{H},\overline{h}), \tag{8}$$

we can deduce the first order averaged equations of motion:

$$\begin{aligned} \dot{\overline{l}} &= \frac{\partial\overline{\mathscr{H}}_0^{(0)}}{\partial\overline{L}} + \epsilon\frac{\partial\overline{\mathscr{H}}_0^{(1)}}{\partial\overline{L}} + \delta\frac{\partial\overline{\mathscr{HB}}_0^{(1)}}{\partial\overline{L}} & \dot{\overline{L}} &= 0, \\ \dot{\overline{g}} &= 0 + \epsilon\frac{\partial\overline{\mathscr{H}}_0^{(1)}}{\partial\overline{G}} + \delta\frac{\partial\overline{\mathscr{HB}}_0^{(1)}}{\partial\overline{G}} \quad \text{and} & \dot{\overline{G}} &= 0, \\ \dot{\overline{h}} &= 0 + \epsilon\frac{\partial\overline{\mathscr{H}}_0^{1}}{\partial\overline{H}} + \delta\frac{\partial\overline{\mathscr{HB}}_0^{(1)}}{\partial\overline{H}} & \dot{\overline{H}} &= \delta\frac{\partial\overline{\mathscr{HB}}_0^{(1)}}{\partial\overline{h}}, \end{aligned} \tag{9}$$

which gives

$$\begin{aligned} \dot{\overline{l}} &= \overline{n} + \epsilon\overline{\lambda}\overline{\eta}(1-3\overline{c}^2) + \delta\overline{\lambda}\overline{\eta}(6\overline{s}^2\cos 2\overline{h}), \\ \dot{\overline{g}} &= 0 + \epsilon\overline{\lambda}(1-5\overline{c}^2) + \delta\overline{\lambda}(-2\cos 2\overline{h})(2\overline{c}^2 - 3\overline{s}^2), \\ \dot{\overline{h}} &= 0 + \epsilon\overline{\lambda}(2\overline{c}^2) + \delta\overline{\lambda}(4\overline{c}\cos 2\overline{h}), \end{aligned} \tag{10}$$

if we define $\overline{\lambda} = (-3\overline{n})/(4\overline{a}^2\overline{\eta}^4)$. Considering only the effect of J_2, Eq. (10) can then be written (in the usual Keplerian elements) as

$$\begin{aligned} \dot{\overline{l}} &= \overline{n} + \epsilon\left[-\frac{3}{4}\frac{\overline{n}}{\overline{a}^2(1-\overline{e}^2)^2}\right]\sqrt{1-\overline{e}^2}(1-3\cos^2\overline{I}), \\ \dot{\overline{g}} &= \epsilon\left[-\frac{3}{4}\frac{\overline{n}}{\overline{a}^2(1-\overline{e}^2)^2}\right](1-5\cos^2\overline{I}), \\ \dot{\overline{h}} &= \epsilon\left[-\frac{3}{4}\frac{\overline{n}}{\overline{a}^2(1-\overline{e}^2)^2}\right](2\cos^2\overline{I}). \end{aligned} \tag{11}$$

[2] Similarly, the expression of $\epsilon\mathscr{W}_1$ also the same as the one given by $(-S_1)$ defined in (15) of Brouwer (1959).

The two formula giving the effect of J_2 only on g and h are well known (Szebehely, 1989, p. 129, Roy, 1968, p. 89, Jupp, 1988, p. 128). The peculiar value of the inclination which makes $\dot{\bar{g}}$ to vanish, known as the *critical inclination* $I_c = 63°26'$, is also well known (Szebehely, 1989, p. 130) and led even to some controversy in the past (Jupp, 1988). These two formula can also be derived from the famous Lagrange's planetary equations (Cook, 1988, p. 56, Kaula, 1966, p. 29).

For the sake of completeness, we give also the first order generator $\mathscr{W}_1^* = \epsilon\, \mathscr{W}_1 + \delta\, \mathscr{W}\mathscr{B}_1$ with:

$$\mathscr{W}_1 = \frac{n}{8\eta^3}[2(1-3c^2)(f-l) + 2e(1-3c^2)\sin(f) - 3es^2\sin(f+2g) - 3s^2\sin(2f+2g) - es^2 \times \sin(3f+2g)] \quad (12)$$

and

$$\mathscr{W}\mathscr{B}_1 = \frac{n}{8\eta^3}[12s^2(f-l)\cos(2h) + 6es^2\sin(f+2h) + 6es^2\sin(f-2h) + 3e(1+c)^2\sin(f+2g+2h) + 3e(1-c)^2\sin(f+2g-2h) + 3(1+c)^2 \times \sin(2f+2g+2h) + 3(1-c)^2\sin(2f+2g-2h) + e(1+c)^2\sin(3f+2g+2h) + e(1-c)^2 \times \sin(3f+2g-2h)]. \quad (13)$$

A lot of symmetries between the pairs ($\mathscr{H}_1^{(0)}$ and $\mathscr{W}_1$) and ($\mathscr{H}\mathscr{B}_1^{(0)}$ and $\mathscr{W}\mathscr{B}_1$) give a lot of confidence in the new first order results in δ.

Once more, for the case $\delta = 0$, we retrieve of course the known results of the isolated effect of J_2: the expression of $\epsilon\, \mathscr{W}_1$ is the same as the one given by $(-S_1)$ defined in (15) of Brouwer (1959).

4.2. New critical inclinations due to the effect of C_{22}

If we now consider both the effect of J_2 and C_{22}, the search for a critical inclination has to be done by solving Eq. (10) for $\dot{\bar{g}} = 0$; which gives as result:

$$\cos^2 \bar{I}_c^* = \frac{\sigma - 6\bar{k}}{5\sigma - 10\bar{k}}, \quad (14)$$

where we use the shortcuts $\sigma = J_2/C_{22} = -\epsilon/\delta$ and $\bar{k} = \cos 2\bar{h}$. The new critical inclination I_c^* is thus defined by a function of $(\sigma, \bar{k})$, which is plotted in Fig. 3 for some values of σ (at least those of Table 1) and all the valid range for $\cos 2\bar{h}$. Let's consider first the following two limiting cases. For $\sigma \to \infty$, we retrieve of course the classical critical inclination $I_c = 63°26'$ of the zonal problem (this is already a good assumption for the case of the Earth, for which $\sigma = 689$). But if we consider the isolated effect of C_{22}, that is the limiting case $\sigma \to 0$, we find a new critical inclination $I_c^* = 39°14'$. Note that $\sigma \geqslant 0$ since $J_2 \geqslant 0$ and $C_{22} \geqslant 0$ by definition of the principal axis of inertia ($A \leqslant B \leqslant C$).

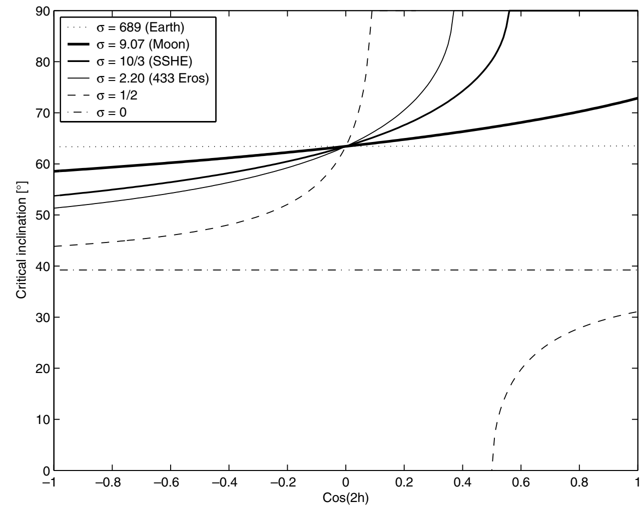

Fig. 3. The effect of C_{22} on the critical inclination I_c^*.

The cases in between these two extremes give quite a big dependence of I_c^* with respect to $\cos 2\bar{h}$. For example, for $\sigma = 9.07$ (lunar satellite), I_c^* ranges roughly from 58° to 72°. The case $\sigma = 10/3$ (SSHE, Synchronous Satellite in Hydrostatic Equilibrium) is quite interesting also, because some high value of $\cos 2\bar{h}$ may never give rise to a critical inclination I_c^* – what we will call a non-critical range. This behavior can be checked by expressing the condition $0 \leqslant \cos^2\bar{I} \leqslant 1$, which gives finally a non-critical range from $\sigma/6$ to σ for $\cos 2\bar{h}$ which is visualized in Fig. 4. If $\sigma \geqslant 6$, then any value of $\cos 2\bar{h}$ gives rise to a critical inclination I_c^*; if $1 \leqslant \sigma \leqslant 6$, then there is an upper non-critical range, while if $0 \leqslant \sigma \leqslant 1$, then there is an intermediate non-critical range. For example, if $\sigma = 1/2$, the intermediate non-critical range

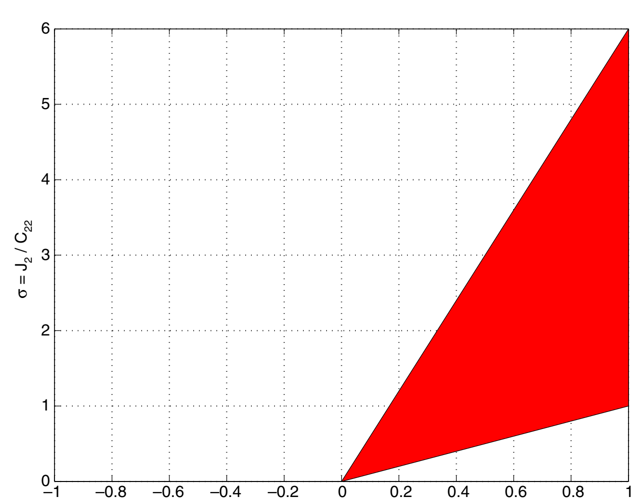

Fig. 4. The non-critical range (shaded area) for $\cos 2\bar{h}$ as a function of σ.

is from $\cos 2\bar{h} = \sigma/6 = 1/12$ to $\cos 2\bar{h} = \sigma = 1/2$ as can be seen in Fig. 3.

4.3. Second order results: in J_2^2 and C_{22}^2 and $J_2 \times C_{22}$

The second order averaged Hamiltonian:

$$\epsilon^2\overline{\mathscr{H}}_0^{(2)} + \epsilon\delta\overline{\mathscr{HM}}_0^{(2)} + \delta^2\overline{\mathscr{HB}}_0^{(2)}, \tag{15}$$

has also been computed. The reader interested in the details should read (De Saedeleer, 2004); we just recall here the final result, with $\mathscr{F} = (3n^2/(64a^2\eta^7))^{-1}$:

$$\begin{aligned}\overline{\mathscr{H}}_0^{(2)} \times \mathscr{F} = {}& [5(s^4 - 8c^4) - 4\eta(1 - 3c^2)^2 - \eta^2(5s^4 - 8c^2) \\ & - 2(1 - 15c^2)\cos(2g)]\end{aligned} \tag{16}$$

and

$$\begin{aligned}\overline{\mathscr{HM}}_0^{(2)} \times \mathscr{F} = {}& [32s^2(5c^2 - 4)\cos(2h) + 48\eta s^2(3c^2 - 1) \\ & \times \cos(2h) - 4e^2s^2(5c^2 + 13)\cos(2h) \\ & - 4e^2(\cos(2g + 2h) + \cos(2g - 2h)) \\ & + 52e^2c(\cos(2g + 2h) - \cos(2g - 2h)) \\ & + 56e^2c^2(\cos(2g + 2h) + \cos(2g - 2h)) \\ & - 60e^2c^3(\cos(2g + 2h) - \cos(2g - 2h)) \\ & - 60e^2c^4(\cos(2g + 2h) + \cos(2g - 2h))]\end{aligned} \tag{17}$$

and also

$$\begin{aligned}\overline{\mathscr{HB}}_0^{(2)} \times \mathscr{F} = {}& [8(-22 + 48c^2 - 10c^4) + 2e^2(-27 \\ & + 78c^2 + 5c^4) - 72\eta s^4(\cos(4h) + 1) \\ & + 10(e^2 - 8)s^4\cos(4h) - 12e^2s^2(3 - 5c^2) \\ & \times \cos(2g) + 30e^2s^2(1 + c)^2\cos(2g + 4h) \\ & + 30e^2s^2(1 - c)^2\cos(2g - 4h)].\end{aligned} \tag{18}$$

Again, symmetries (especially by the way of the tracer η) give good confidence in the new second order averaged results in δ.

We retrieve also some known results for the effect of J_2^2 only ($\delta = 0$): the expression of $\epsilon^2\overline{\mathscr{H}}_0^{(2)}$ is the same as the one given by $(-2F_2^*)$ defined in (29) of Brouwer (1959). Moreover, the generator in ϵ^2 has also been computed and validated: the exact equivalence with the Kozai's S_2 (given by equation (3.2) of Kozai (1962, p. 448)) has been established elsewhere, using the relationships of (Shniad, 1970) for the correspondence between generators of von Zeipel (S_i) and the ones of Lie ($\mathscr{W}_i$).

A critical issue by making such computations is the rapid growth of the number of terms in the series. Needless to say, quite a big work of simplification has to be made, especially because of the redundancy ($\eta \leftrightarrow e$) and ($c \leftrightarrow s$) of the set (4). Numerical accuracy considerations are also required (as we are working in double precision).

5. Conclusions

We have achieved a.o. second order results for the combined effect of J_2 and C_{22} using the Lie Transform. The averaged Hamiltonians and the generators for the first and second order effects of J_2 have been validated by comparison with the results of Brouwer (1959) and Kozai (1962).

An important first order result is that the critical inclination of $I_c = 63°26'$ of the zonal problem may be strongly affected by the C_{22} coefficient and by the value of the longitude of the ascending node h; I_c^* may be in the whole range from 0° to 90°, depending on the value of the ratio $\sigma = J_2/C_{22}$.

The study of a more complete model of the lunar artificial satellite is ongoing and is intended to be published in a forthcoming paper.

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