Celestial Mechanics and Dynamical Astronomy (2006) 95:407–423
DOI 10.1007/s10569-006-9029-6

ORIGINAL ARTICLE

Analytical theory of a lunar artificial satellite with third body perturbations

Bernard De Saedeleer

Received: 15 November 2005 / Revised: 23 March 2006 /
Accepted: 7 May 2006 / Published online: 15 August 2006

Abstract We present here the first numerical results of our analytical theory of an artificial satellite of the Moon. The perturbation method used is the Lie Transform for averaging the Hamiltonian of the problem, in canonical variables: short-period terms (linked to l, the mean anomaly) are eliminated first. We achieved a quite complete averaged model with the main four perturbations, which are: the synchronous rotation of the Moon (rate $n_{☾}$), the oblateness J_2 of the Moon, the triaxiality C_{22} of the Moon ($C_{22} \approx J_2/10$) and the major third body effect of the Earth (ELP2000). The solution is developed in powers of small factors linked to these perturbations up to second-order; the initial perturbations being sorted ($n_{☾}$ is first-order while the others are second-order). The results are obtained in a closed form, without any series developments in eccentricity nor inclination, so the solution apply for a wide range of values. Numerical integrations are performed in order to validate our analytical theory. The effect of each perturbation is presented progressively and separately as far as possible, in order to achieve a better understanding of the underlying mechanisms. We also highlight the important fact that it is necessary to adapt the initial conditions from averaged to osculating values in order to validate our averaged model dedicated to mission analysis purposes.

Keywords Lunar artificial satellite · Third body · Lie · Hamiltonian · C_{22} · Earth

1 Introduction

We reached a corner stone in the development of our analytical theory of a lunar artificial satellite. For the first time, we achieved a complete averaged model with the main four perturbations, which are: the synchronous rotation of the Moon (rate $n_{☾}$),

B. De Saedeleer (✉)
Département de Mathématique, University of Namur, Rempart de la Vierge 8,
B-5000 Namur, Belgium
e-mail: Bernard.DeSaedeleer@fundp.ac.be

the oblateness J_2 of the Moon, the triaxiality C_{22} of the Moon ($C_{22} \approx J_2/10$) and the major third body effect of the Earth (ELP2000). Our goal is to build an averaged model for mission analysis purposes, and not to make any orbit determination.

In some previous paper (De Saedeleer 2004), we developed the perturbations in J_2 and C_{22}, and averaged them up to order J_2^2, C_{22}^2 and $J_2 \times C_{22}$. In another one (De Saedeleer and Henrard 2005), we detailed the development of the third body (Earth) perturbation by making use of the lunar theory ELP2000 (Chapront-Touzé and Chapront 1991).

Now, in this paper, we present our latest new results: the averaging of that third body perturbation and hence the building of a quite complete averaged model. Moreover, we present also here the first numerical integrations which come along with that averaged model, and which validate our analytical theory.

The perturbation method used is the Lie Transform for averaging the Hamiltonian of the problem, in canonical variables: short-period terms (linked to l, the mean anomaly) are eliminated first. The solution is developed in powers of small factors linked to these perturbations. The initial perturbations are sorted in such a way that $n_☾$ is first-order while the others are second-order. The averaging process is done up to second-order, which then means that the first-order effect of the perturbations is in fact captured.

Of course, the determination of the motion of a lunar satellite has already drawn some attention in the past (Oesterwinter 1970; Milani and Knežević 1995; Steichen, 1998a, b). So, we could extensively cross-check some of our results with the literature (see Sect. 3), but we also have gone a step further in the understanding of the dynamics.

It turns out that the problem of the lunar orbiter is quite interesting because its dynamics is different from the one of an artificial satellite of the Earth, by at least two aspects: the C_{22} lunar gravity term is only 1/10 of the J_2 term and the third body effect of the Earth on the lunar satellite is much larger than the effect of the Moon on a terrestrial satellite. So we have to account at least for these larger perturbations.

Our goal is not to go to very high order in J_2, nor to add many harmonics, while it could be done easily in principle, for example by addressing the complete zonal problem (De Saedeleer 2005); we rather want to highlight the main parameters affecting the dynamics, hence we deliberately choose to restrict the study to the aforementioned four main perturbations.

The structure of this paper is as follows. The geometry, variables and perturbations are described in Sect. 2; the averaged Hamiltonian is given in Sect. 3; the numerical integrations are introduced in Sect. 4; the effect of J_2 is adressed in Sect. 5; the additional effect of C_{22} and of ($n_☾$+ the Earth) is discussed in Sects. 6 and 7, respectively; the adaptation of the initial conditions from averaged to osculating values is discussed in Sect. 8 (with a detailed example given in Appendix); we then conclude in Sect. 9.

2 Geometry, variables and perturbations

We use here the canonical method of the Lie Transform (Deprit 1969). In order to keep the Hamiltonian formalism, it is required to work in canonical variables; we choose the classical Delaunay variables (l, g, h, L, G, H) defined as:

$$
\begin{aligned}
l &= u - e \sin u, \qquad & L &= \sqrt{\mu a}, \\
g &= \omega, & G &= \sqrt{\mu a(1 - e^2)}, \\
h &= \Omega, & H &= \sqrt{\mu a(1 - e^2)} \cos I,
\end{aligned}
\tag{1}
$$

where $(a, e, I, \omega, \Omega)$ are the keplerian elements, $\mu = GM_{☾}, l$ and u are the mean and eccentric anomaly, respectively. In these variables, the unperturbed potential is simply written $-\mu^2/(2L^2)$.

Now we have to write all the perturbations in these variables and in an inertial frame; that is to say with respect to a constant direction in space. The inertial frame (x, y, z) is chosen so that its origin is taken at the center of the Moon and so that the (x, y) plane is the lunar equatorial plane (see Fig. 1).

In order to be able to use the expressions of the spherical harmonics for the potential, we first have to define spherical coordinates (r, λ', ϕ), so that the longitude of the satellite λ' starts from the x axis in the equatorial plane, the latitude ϕ being defined as the deviation from the (x, y) plane. Within that inertial frame, the perturbative potentials in J_2 and C_{22} may be written $(V_{20} - V_{22})$, with:

$$
V_{20} = \frac{\mu}{r}\left(\frac{R}{r}\right)^2 J_2 P_{20}(\sin\phi), \quad \text{where } P_{20}(x) = \frac{1}{2}(3x^2 - 1), \tag{2}
$$

$$
V_{22} = \frac{\mu}{r}\left(\frac{R}{r}\right)^2 C_{22} P_{22}(\sin\phi)\cos(2(\lambda' - \lambda_{22})), \quad \text{where } P_{22}(x) = 3(1 - x^2), \tag{3}
$$

where R is the equatorial radius of the Moon ($R \approx 1,738\,\text{km}$); P_{20} and P_{22} being the Legendre Associated Functions. We can partially translate their argument $(\sin\phi)$ into Delaunay variables by the way of the spherical trigonometry (see Fig. 1, where the plane of the orbit is at an inclination I): $\sin\phi = \sin I \sin(f + g)$. We then have:

$$
V_{20} = +J_2 R^2(\mu r^{-3})\left(1 - 3c^2 - 3s^2\cos(2f + 2g)\right)/4. \tag{4}
$$

But the coefficient C_{22} makes the longitude λ' to appear in addition to the latitude ϕ. The spherical harmonics being defined with respect to the main axis of inertia of the attracting body, we had to define λ_{22} as the longitude of the lunar longest meridian (minimum inertia). This angle makes the Hamiltonian to be time-dependent, since $\lambda_{22} = \lambda_\oplus$ travels at the rate of the synchronous rotation which is $\dot{\lambda}_\oplus = n_{☾}$. In order to

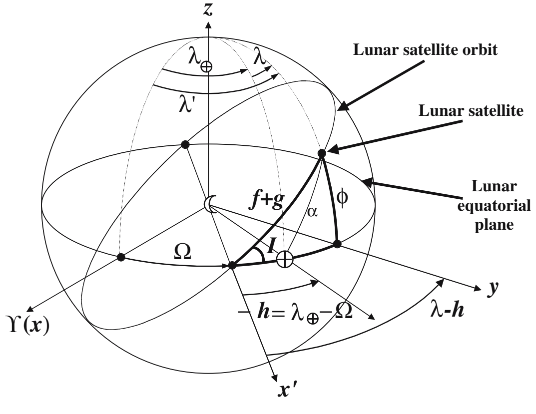


Fig. 1 Simplified selenocentric sphere. The center of the Moon is taken as the origin; the lunar equatorial plane is taken as the (x, y) plane and $\lambda_\oplus$ is the longitude of the Earth

eliminate this dependency, we will work in a rotating system whose x' axis now passes through the Earth; we then define new longitudes with respect to $\lambda_\oplus$: $\lambda = \lambda' - \lambda_\oplus$ and we redefine also $h = \Omega - \lambda_\oplus$ (the angles always appear in that combination). A term $(-n_☾ H)$ has to be added to the Hamiltonian in order to take this rotation into account.

With that definition of λ, we have also now $V_{22} = +C_{22}R^2\mu r^{-3}P_{22}(\sin\phi)\cos(2\lambda)$. Once again, the factor $\cos(2\lambda)$ can be partially translated into Delaunay variables by the same way of the spherical trigonometry, which gives finally:

$$\begin{aligned} V_{22} \quad = \quad & +C_{22}R^2(\mu r^{-3})3\big\{2s^2\cos(2h) + (c+1)^2\cos(2f+2g+2h) \\ & + (c-1)^2\cos(2f+2g-2h)\big\}/4. \end{aligned} \tag{5}$$

At this stage, there remains in (4) and (5) only r and f to be expressed as a function of (l, g, h) in order to be able to apply a perturbation method. It turns out that the functions $r = r(l, g, h)$ and $f = f(l, g, h)$ cannot be expressed in a closed form, and that one usually falls back at this point into series development in the eccentricity. We would like to avoid this, at least for the following reasons: the results would be much less compact, hence a lack of ease to interpret the results; moreover they would no longer be valid for higher values of the eccentricity.

So we prefer to use the following set of auxiliary variables $(\xi, f, g, h, a, n, e, \eta, s, c)$ in order to describe the position of the lunar satellite:

$$\begin{aligned} &\xi = \tfrac{a}{r} = \frac{1+e\cos f}{1-e^2} = \frac{1}{1-e\cos u}, && f, \\ &a = \frac{L^2}{\mu}, && n = \frac{\mu^2}{L^3}, \\ &e = \sqrt{1-\left(\frac{G}{L}\right)^2} && \eta = \sqrt{1-e^2} = \frac{G}{L}, \\ &s = \sin I = \sqrt{1-\left(\frac{H}{G}\right)^2} && c = \cos I = \frac{H}{G}, \\ &g && h, \end{aligned} \tag{6}$$

where f is the true anomaly.

This set has a major advantage: it leads to formulae in closed form with respect to the eccentricity and inclination. The only two drawbacks are (1) that it is redundant $(e^2 + \eta^2 = 1; c^2 + s^2 = 1)$ and (2) that we need to perform partial derivatives of them with respect to the canonical variables; but it is not too heavy a task. We have for example: $\frac{dA}{dl} = \frac{\partial A}{\partial l} + \frac{\partial A}{\partial \xi}\frac{\partial \xi}{\partial l} + \frac{\partial A}{\partial f}\frac{\partial f}{\partial l}$. This choice of variables and all the partial derivatives of them with respect to the canonical variables (l, g, h, L, G, H) have already been described in De Saedeleer and Henrard (2005). The computation of the partial derivatives themselves requires some caution, but is not too complicated; use has to be made of the Kepler equation $(l = E - e\sin E)$, which links the anomalies. We have for example $\partial f/\partial G = -\sin f(1 + \xi\eta^2)/(\eta n a^2 e)$ and also $\partial f/\partial l = \xi^2\eta$, a quantity which plays an important role, since it will allow to switch the integration from l to f.

We can then rewrite the complete Hamiltonian in this set of variables (6). Note that the factor (μr^{-3}) appearing in (4) and (5) is simply written $\xi^3 n^2$. The unperturbed potential is $\mathcal{H}_0^{(0)} = -\mu^2 L^{-2}/2$, while we sort the four perturbations by their order of magnitude. The mean motion of the Moon $n_☾$ is about 0.23 rad/day ($n_☾ = 2\pi/T_☾$ with the sidereal rotation period of the Moon $T_☾ \approx 27.32$ days). For a typical lunar orbit (altitude around 500 km), the lunar satellite has a period of 2.64 hours; n is then about 57.12 rad/day. If we choose that frequency as a unit, $n_{☾}$ is about 4×10^{-3}, while the J_2 and the C_{22} terms are of order 10^{-4} and 10^{-5}, respectively. Additionally, we have the relationship $\gamma = -\mu_{\oplus} a_{\oplus}^{-3} = -n_{☾}^2 M_{\oplus}/(M_{\oplus} + M_{☾}) \approx -n_{☾}^2$, so that γ is indeed quite very exactly of second-order with respect to $n_{☾}$.

In summary, we may put the biggest perturbation ($n_{☾}$) at first-order, and the other lower ones all at second-order, which gives the following final arrangement:

$$\mathcal{H}^{(0)} = \mathcal{H}_0^{(0)} + \mathcal{H}_1^{(0)} + \epsilon\, \mathcal{H}_2^{(0)} + \delta\, \mathcal{HB}_2^{(0)} + \gamma\, \mathcal{HE}_2^{(0)} \tag{7}$$

with:

$$\mathcal{H}_1^{(0)} = -n_{☾} H, \tag{8}$$

$$\mathcal{H}_2^{(0)} = \xi^3 n^2 \left(1 - 3c^2 - 3s^2 \cos(2f + 2g)\right) \Big/ 4, \tag{9}$$

$$\begin{aligned}\mathcal{HB}_2^{(0)} = {} & 3\xi^3 n^2 \{2s^2 \cos(2h) + (c+1)^2 \cos(2f + 2g + 2h) \\ & + (c-1)^2 \cos(2f + 2g - 2h)\}/4,\end{aligned} \tag{10}$$

$$\mathcal{HE}_2^{(0)} = a_{\oplus}^3 r_{\oplus}^{-3} a^2 \xi^{-2} P_{20}(\cos\alpha) \tag{11}$$

and with $\epsilon = J_2 R^2, \delta = -C_{22} R^2, \gamma = -\mu_{\oplus} a_{\oplus}^{-3}$, and where α is the angle between the Earth and the lunar satellite.

The computation of $\cos\alpha = \vec{r} \cdot \vec{r}_{\oplus}/(r r_{\oplus})$ requires the knowledge of the direction of the Earth from the Moon $\vec{A}_{\oplus} = (A_{\oplus}, B_{\oplus}, C_{\oplus})$. For this, we use the lunar theory ELP2000 (Chapront-Touzé and Chapront 1991), which gives the opposite direction, in spherical coordinates. In that theory, the position of the Moon is described by a series of periodic functions mainly of the fundamental arguments L^*, D, l', l^*, F; from which we take the leading terms. Let's recall that L^* is the secular part of the mean longitude of the Moon referred to the mean dynamical ecliptic and equinox of date, D is the secular part of the difference between the mean longitude of the Moon and the geocentric mean longitude of the Sun, l' is the secular part of the geocentric mean anomaly of the Sun, l^* is the secular part of the mean anomaly of the Moon, F is the secular part of the difference between the mean longitude of the Moon and of the longitude of its ascending node on the mean ecliptic of date.

As already mentioned, the development of these perturbations have already been described elsewhere in deeper details (see De Saedeleer 2004 for ϵ and δ, and De Saedeleer and Henrard 2005 for γ and $n_{☾}$). We just give here in Table 1 the very first terms ($|\text{Coefficient}| > 0.1$) of the second-order perturbations.

In that table, we immediately recognize (9) and (10), while we can rewrite the part corresponding to (11) in full:

$$\begin{aligned}\mathcal{HE}_2^{(0)} = {} & a^2 \xi^{-2} \Big[-0.12466(1 - 3c^2) + 0.37225\, s^2 \cos(2(h - L^*)) \\ & + 0.37397\, s^2 \cos(2(f + g)) + 0.18612 \Big\{(1 + c)^2 \cos(2(f + g + h - L^*)) \\ & + (1 - c)^2 \cos(2(f + g - h + L^*))\Big\}\Big].\end{aligned} \tag{12}$$

We use then the Lie Transform (Deprit 1969) as canonical perturbation method, with the four parameters $(n_{☾}, \epsilon, \delta, \gamma)$, all gathered in the Lie triangle (see Fig. 2), which is filled by the recursive formula $\mathcal{H}_i^{(j)} = \mathcal{H}_{i+1}^{(j-1)} + \sum_{k=0}^{i} C_i^k \left(\mathcal{H}_{i-k}^{(j-1)}; \mathcal{W}_{k+1}\right)$; note

Table 1 The $\epsilon\,\mathcal{H}_0^{(2)} + \delta\,\mathcal{HB}_0^{(2)} + \gamma\,\mathcal{HE}_0^{(2)}$ series (12 terms)

	f	g	h	L^*	ξ	a	n	c	s	δ	ϵ	γ	$1+c$	$1-c$	Coefficient
cos	0	0	0	0	3	0	2	0	0	0	1	0	0	0	0.25000D + 00
cos	0	0	0	0	3	0	2	2	0	0	1	0	0	0	−0.75000D + 00
cos	2	2	0	0	3	0	2	0	2	0	1	0	0	0	−0.75000D + 00
cos	0	0	0	0	−2	2	0	0	0	0	0	1	0	0	−0.12466D + 00
cos	0	0	0	0	−2	2	0	2	0	0	0	1	0	0	0.37398D + 00
cos	0	0	2	−2	−2	2	0	0	2	0	0	1	0	0	0.37225D + 00
cos	2	2	0	0	−2	2	0	0	2	0	0	1	0	0	0.37397D + 00
cos	2	2	2	−2	−2	2	0	0	0	0	0	1	2	0	0.18612D + 00
cos	2	2	−2	2	−2	2	0	0	0	0	0	1	0	2	0.18612D + 00
cos	0	0	2	0	3	0	2	0	2	1	0	0	0	0	0.15000D + 01
cos	2	2	2	0	3	0	2	0	0	1	0	0	2	0	0.75000D + 00
cos	2	2	−2	0	3	0	2	0	0	1	0	0	0	2	0.75000D + 00

$$
\begin{array}{ccccc}
\mathcal{H}_0^{(0)} = -\frac{\mu^2}{2L^2} & & & & \\
\mathcal{H}_1^{(0)} = -n_{☾} H & \mathcal{H}_0^{(1)} & & & \\
\epsilon\,\mathcal{H}_2^{(0)} + \delta\,\mathcal{HB}_2^{(0)} + \gamma\,\mathcal{HE}_2^{(0)} & \mathcal{H}_1^{(1)} & \mathcal{H}_0^{(2)} & & \\
0 & \mathcal{H}_2^{(1)} & \mathcal{H}_1^{(2)} & \mathcal{H}_0^{(3)} & \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

Fig. 2 Our specific Lie triangle, with the first ($n_{☾}$) and second (ϵ, δ, γ) order perturbations

that an appropriate scaling is done to the perturbations in order to fulfill the scheme $\mathcal{H} = \sum_{i\geq 0} \frac{\epsilon^i}{i!}\mathcal{H}_i^{(0)}$. We write $\mathcal{H}_0^{(i)}$ as $\overline{\mathcal{H}}_0^{(i)}$ in order to remember that the fast angle l has been eliminated; we always put the periodic part in the generator $\mathcal{W}_i$.

3 Averaged Hamiltonian and symbolic manipulation software MM

In order to make the symbolic computations of the averaged theory, we used a specific FORTRAN code called the MM, standing for "Moon's series Manipulator", which has been developed at our university, and which is dedicated to algebraic manipulations. In this tool, each expression is given by a series of linear trigonometric functions, with polynomial coefficients. The property of linearity will make the integrations very straightforward. An example of such a series has been given in Table 1. The computations are done in double precision but we display only five digits, which is sufficient for the purposes of this article.

It is of course impossible to give a comprehensive view of all the results in the scope of this paper, since the series may contain a lot of terms, but we give however explicitly some of them here, and we comment the others. For the first-order, as $\mathcal{H}_1^{(0)}$ is already independent of l, we have $\overline{\mathcal{H}}_0^{(1)} = \mathcal{H}_1^{(0)} = -n_{☾} H$ and $\mathcal{W}_1 = 0$. For the second-order,

Table 2 The $\epsilon\overline{\mathcal{H}}_0^{\omega} = \overline{\mathcal{H}}_0^{(1)} + \epsilon\,\mathcal{B}_0^{(2)} + \delta\,\overline{\mathcal{HB}}_0^{(2)} + \gamma\,\overline{\mathcal{HE}}_0^{(2)}$ series (13 terms)

	$\bar{g}$	$\bar{h}$	L^*	$\bar{a}$	$\bar{n}$	$\bar{e}$	$\bar{\eta}$	$\bar{c}$	$\bar{s}$	δ	$n_{☾}$	ϵ	γ	$1+\bar{c}$	$\bar{H}$	$1-\bar{c}$	Coefficient
cos	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	−0.10000D + 01
cos	0	0	0	0	2	0	−3	0	0	0	0	1	0	0	0	0	0.25000D + 00
cos	0	0	0	0	2	0	−3	2	0	0	0	1	0	0	0	0	−0.75000D + 00
cos	0	0	0	2	0	0	0	0	0	0	0	0	1	0	0	0	−0.12466D + 00
cos	0	0	0	2	0	0	0	2	0	0	0	0	1	0	0	0	0.37398D + 00
cos	0	0	0	2	0	2	0	0	0	0	0	0	1	0	0	0	−0.18699D + 00
cos	0	0	0	2	0	2	0	2	0	0	0	0	1	0	0	0	0.56096D + 00
cos	0	2	−2	2	0	0	0	0	2	0	0	0	1	0	0	0	0.37225D + 00
cos	0	2	−2	2	0	2	0	0	2	0	0	0	1	0	0	0	0.55837D + 00
cos	2	0	0	2	0	2	0	0	2	0	0	0	1	0	0	0	0.93493D + 00
cos	2	2	−2	2	0	2	0	0	0	0	0	0	1	2	0	0	0.46531D + 00
cos	2	−2	2	2	0	2	0	0	0	0	0	0	1	0	0	2	0.46531D + 00
cos	0	2	0	0	2	0	−3	0	2	1	0	0	0	0	0	0	0.15000D + 01

we have:

$$\epsilon\mathcal{H}_0^{(2)} + \delta\mathcal{HB}_0^{(2)} + \gamma\mathcal{HE}_0^{(2)} = \epsilon\mathcal{H}_2^{(0)} + \delta\mathcal{HB}_2^{(0)} + \gamma\mathcal{HE}_2^{(0)} + \left(\mathcal{H}_0^{(0)};\mathcal{W}_2\right) \tag{13}$$

and we choose:

$$\epsilon\overline{\mathcal{H}}_0^{(2)} + \delta\overline{\mathcal{HB}}_0^{(2)} + \gamma\overline{\mathcal{HE}}_0^{(2)} = \frac{1}{2\pi}\int_0^{2\pi}\left(\epsilon\mathcal{H}_2^{(0)} + \delta\mathcal{HB}_2^{(0)} + \gamma\mathcal{HE}_2^{(0)}\right)\mathrm{d}l \tag{14}$$

while $(\mathcal{H}_0^{(0)};\mathcal{W}_2)$ reduces to $n\frac{\partial\mathcal{W}_2}{\partial l}$, which has then to be integrated with respect to l.

The integration of the terms in (ϵ,δ,γ), which is in fact rather a first-order averaging, may be performed in closed form quite easily by using techniques described in De Saedeleer (2004) for (ϵ,δ) using ξ and f, and in Jefferys (1971) for γ, which uses additionally u. Higher orders may be achieved by the same way, provided we are able to compute the integrals. The third-order would contain the combinations of perturbation parameters $(\epsilon n_{☾},\delta n_{☾},\gamma n_{☾})$ and the fourth-order $(\epsilon^2,\delta^2,\gamma^2,\epsilon\delta,\epsilon\gamma,\delta\gamma,\epsilon n_{☾}^2,$ $\delta n_{☾}^2,\gamma n_{☾}^2)$.

In this article, we mainly focus on the first-order effects (ϵ,δ,γ), while some higher order effects (like $\epsilon^2,\delta^2,\epsilon\delta$) have already been described in De Saedeleer (2004). The second-order averaged Hamiltonian (in ϵ,δ,γ) is given in Table 2, from which we can derive the averaged equations of motion.

It can be rewritten in full as follows:

$$\begin{aligned}\overline{\mathcal{H}} \;=\; & \overline{\mathcal{H}}_0^{(1)} + \epsilon\overline{\mathcal{H}}_0^{(2)} + \delta\overline{\mathcal{HB}}_0^{(2)} + \gamma\overline{\mathcal{HE}}_0^{(2)} = -n_{☾}\bar{H} + \epsilon\frac{\bar{n}^2}{4\bar{\eta}^3}(1-3\bar{c}^2) \\ & + \delta\frac{3\bar{n}^2}{2\bar{\eta}^3}\left(\bar{s}^2\cos(2\bar{h})\right) + \gamma\bar{a}^2\Big[(1-3\bar{c}^2)(-0.12466-0.18699\bar{e}^2) \\ & + 0.37225\,\bar{s}^2\cos(2(\bar{h}-\bar{L}^*)) + 0.55837\bar{s}^2\bar{e}^2\cos(2(\bar{h}-\bar{L}^*)) + 0.93493\,\bar{s}^2\bar{e}^2\cos(2\bar{g}) \\ & + 0.46531\,\bar{e}^2\left\{(1+\bar{c})^2\cos(2(\bar{g}+\bar{h}-\bar{L}^*)) + (1-\bar{c})^2\cos(2(\bar{g}-\bar{h}+\bar{L}^*))\right\}\Big].\end{aligned} \tag{15}$$

Several validations have been carried out in some previous papers, mainly the effect of J_2. It is not the purpose of this paper to give these validations in details, but we give here however an overview. For the first-order (ϵ): the averaged Hamiltonian and the generator are the same as the results of Brouwer (1959). For the second-order (ϵ^2): the averaged Hamiltonian is the same as (Brouwer 1959), while the generator is equivalent to the generator S_2 given by Eq. 3.2 of Kozai (1962), as it has been shown in De Saedeleer and Henrard (2005), which uses the relationships of Shniad (1970) for the correspondence between generators of von Zeipel (S_i) and the ones of Lie ($\mathcal{W}_i$). We just remind here the expression of the averaged Hamiltonian in ϵ^2:

$$\epsilon^2\overline{\mathcal{H}}_0^{(4)} = \frac{\epsilon^2 3n^2}{128a^2\eta^7}\Big[5(s^4-8c^4)-4\eta(1-3c^2)^2 - \eta^2(5s^4-8c^2)-2e^2s^2(1-15c^2)\cos(2g)\Big]. \qquad (16)$$

4 Numerical integrations

In the following sections, we will investigate numerically the several effects gradually in order to see more clearly the effect of each additional perturbation: ϵ alone in Sect. 5, $(\epsilon+\delta)$ in Sect. 6, $(\epsilon+\delta+n_{☾})$ and $(\epsilon+\delta+n_{☾}+\gamma)$ in Sect. 7. The averaged equations of motion deduced from (15) were integrated numerically; an improved version of the Burlish–Stoer subroutine (Press et al. 1986) has been used. The following set of numerical values for the averaged initial conditions has been chosen:

$$\bar{l}_0 = 10\,\text{rad},\quad \bar{g}_0 = 1\,\text{rad},\quad \bar{h}_0 = 2\,\text{rad},\quad \bar{a}_0 = 3{,}000\,\text{km},\quad \bar{e}_0 = 0.2,\quad \bar{i}_0 = 30\,\text{deg}. \qquad (17)$$

We also took $\mu_{☾} = 3.66\times10^{13}\,\text{km}^3/\text{day}^2$ and $\mu_{\oplus} = 81.3\mu_{☾}$; the period of the satellite is about 4.1 hours for $a = 3{,}000$ km. For the perturbation parameters, we took: $\epsilon = 613.573\,\text{km}^2$; $\delta = -67.496\,\text{km}^2$; $n_{☾} = 0.230$ rad/day; $\gamma = -0.05214$ rad/day^2. We can easily select an isolated effect by putting the other parameters to zero.

5 Effect of J_2 alone

The effect of J_2 alone (first- and second-order) is shown in Fig. 3. At first order, (a, e, i) remain constant while the angles g and h do precess, with periods of approximately 3 and 5 years, respectively. These rates are consistent with the two well-known classical formula, given, i.e. in Szebehely (1989); Roy (1968); Jupp (1988):

$$\dot{\omega} = (3n/2)J_2(R/p)^2(2-(5/2)\sin^2 i), \qquad (18)$$

$$\Omega = (-3n/2)J_2(R/p)^2\cos i, \qquad (19)$$

with $p = a(1-e^2)$. The associated peculiar value of the inclination which makes $\dot{\bar{\omega}}$ to vanish, known as the *critical inclination* $I_c = 63°26'$, is quite famous (Szebehely 1989). Note that the rate of precession of the elements of the orbit of a lunar satellite is much lower than in the case of artificial satellites of the Earth, since the J_2 of the Moon is lower.

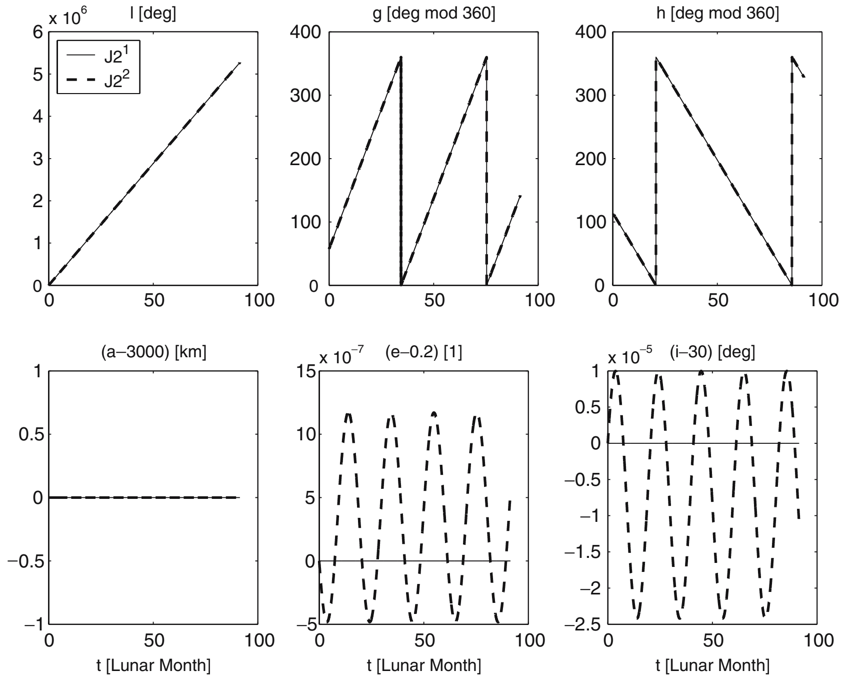


Fig. 3 The effect of J_2 alone ($\epsilon \neq 0, \delta = 0 = n_{☾} = \gamma$): integration of the averaged models (15) and (16) for the first- and second-order effect, respectively; in both cases the initial conditions are (17)

At second-order in J_2 (integration of the averaged equations of motion deduced from (16)), e and i start to oscillate, since the averaged Hamiltonian contains a factor like $\cos(2g)$, hence $\dot{G} \neq 0$. The period is then half of 3 years, (around 18 lunar monthes, well noticeable in Fig. 3), but the amplitude of the oscillations are however, small: 1.654×10^{-6} for e, and 3.420×10^{-5} deg for i.

6 Combined effect of J_2 and C_{22}

We come back to the first-order in J_2 now, where (a, e, i) were constant. If we add the perturbation in C_{22}, the angle h enters the game, by a factor like $\cos(2h)$ this time, so that now $\dot{H} \neq 0$, hence i start to oscillate (but still not e, since $\dot{G} = 0$). Now the amplitudes are very significant, since it is a first-order effect; in our numerical example (plotted in Fig. 4), i oscillates roughly from 29 to 37 deg. The period is half of 5 years (around 35 lunar monthes, well noticeable in Fig. 4).

The introduction of C_{22} has another consequence: it modifies quite significantly the classical critical inclination $I_c = 63°26'$ to new critical inclinations I_c^*, as has been shown in De Saedeleer and Henrard (2006). In the case of the Moon, I_c^* may lie in the range 58–72 deg.

7 Additional effect of $n_{☾}$ and of the Earth

The effect of $n_{☾}$ and of the Earth is shown in Fig. 5. Let's first look at the dashed curves, labelled "without Earth". This case corresponds to the effect of the perturbations

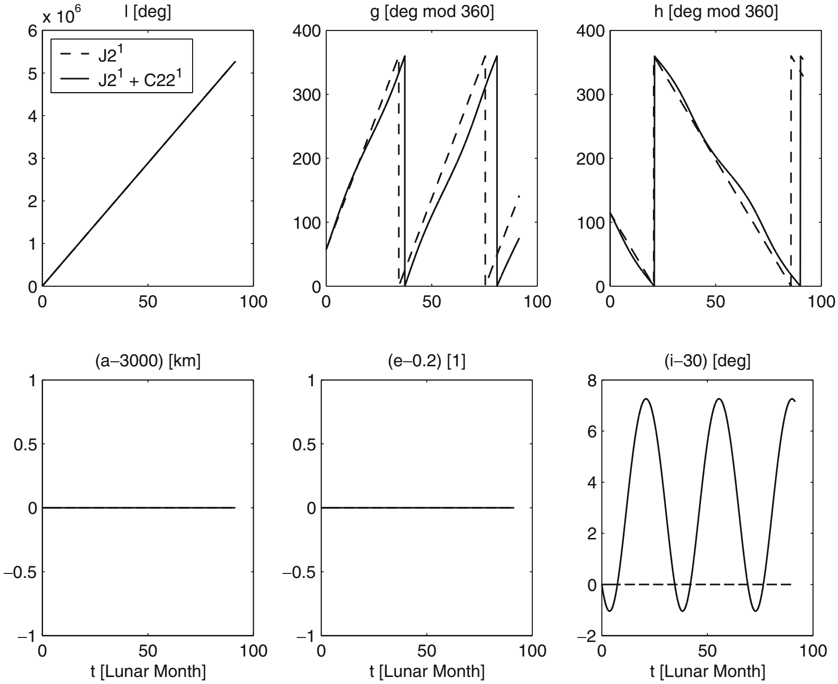


Fig. 4 The combined effect of J_2 and C_{22} ($\epsilon \neq 0, \delta \neq 0, n_{☾} = 0 = \gamma$): integration of the averaged model (15) for the first-order effects; the initial conditions are (17)

($J_2 + C_{22} + n_{☾}$). So, in a first step, only $n_{☾}$ has been added with respect to Sect. 6: the consequence is that the angle h now rotates more quickly: the period is the month (the synchronous rotation) instead of 5 years, hence the inclination also vary, now with a half-month period; the amplitude is quite small: about 0.05 deg.

We then added the effect of the Earth, by considering in a first approximation only a few terms of the third body perturbation (see Table 1). We see that the inclination is now modulated by a period of about 1.2 years with larger amplitude (0.5 deg), coming from a factor like $\sin(2g)$ in $\dot{G}$, with a period of 2.4 years for g. More significant is the variation of the eccentricity, which was constant until now. The eccentricity starts to oscillate, with a fourth month period and quite small amplitudes; but the same long-term modulation as for i also appears (a period of about 1.2 years with larger amplitudes of about 0.02).

It is nowadays known how the stability of a lunar satellite can be strongly affected by the presence of the Earth, especially for higher orbits, while the J_2 effect is stabilizing. The fact that higher orbits are more unstable than lower ones is quite counterintuitive and can lead to surprises. On the other hand, very low orbits are even surprising, since they may also become unstable under the influence of other (odd) gravity harmonics (Knežević and Milani 1998), as was learned the hard way in the past with the crash of Apollo 16 subsatellite only 35 days after its release (Konopliv et al. 1993).

Of course, the dynamics is still strongly dependent on the initial conditions. The eccentricity may sometimes become so high that the satellite crashes on the Moon, as it is the case for polar orbiters. We made a parametric study of the lifetimes of lunar

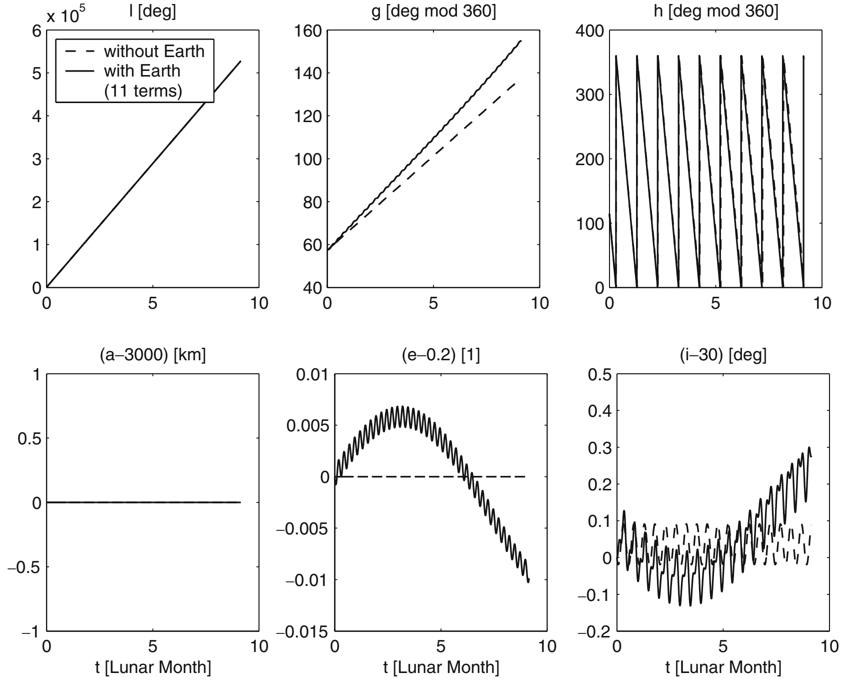


Fig. 5 The additional effect of $n_{☾}$ and of the Earth: integration of the averaged model (15) for the first-order effects; we took in both cases ($\epsilon \neq 0, \delta \neq 0, n_{☾} \neq 0$) and the initial conditions (17). Then we took once $\gamma = 0$ (*dashed line*: without the effect of the Earth) and once $\gamma \neq 0$ (*solid line*: with the effect of the Earth)

polar orbiters and the results were in agreement with (Steichen 1998b; Liu and Wang 2000). The present theory, when it will be completely averaged, will allow that kind of very rapid mission analysis for a wide range of initial conditions.

The effect of the number of terms taken for the Earth is shown in Fig. 6. We integrated the averaged model (15) for the first-order effects, with ($\epsilon \neq 0, \delta \neq 0, n_{☾} \neq 0, \gamma \neq 0$) and the initial conditions (17). The terrestrial perturbation contained (1) once 11 terms (accuracy 10^{-6}) and (2) once 350 terms (accuracy 10^{-9}); we then plot the difference (1)–(2) in each of the elements, which is of order 10^{-3}. One conclude that the main trend was already given by the leading terms that were given in Table 1, but that considering more terms can give a somewhat more accurate description.

8 Osculating versus averaged initial conditions

In this section, we present a qualitative validation of the averaging process: we compare the averaged motion (integration of the averaged model (15)) to the osculating one (integration of the osculating model (7)). For illustration purposes, we take again our simple example of Sect. 5: first-order effect in J_2 alone ($\epsilon \neq 0, \delta = 0 = n_{☾} = \gamma$), where averaged (a, e, i) were constant.

If we do not pay attention, some surprises can arise. For instance, if we take the same initial conditions for both the osculating motion and the averaged motion, the

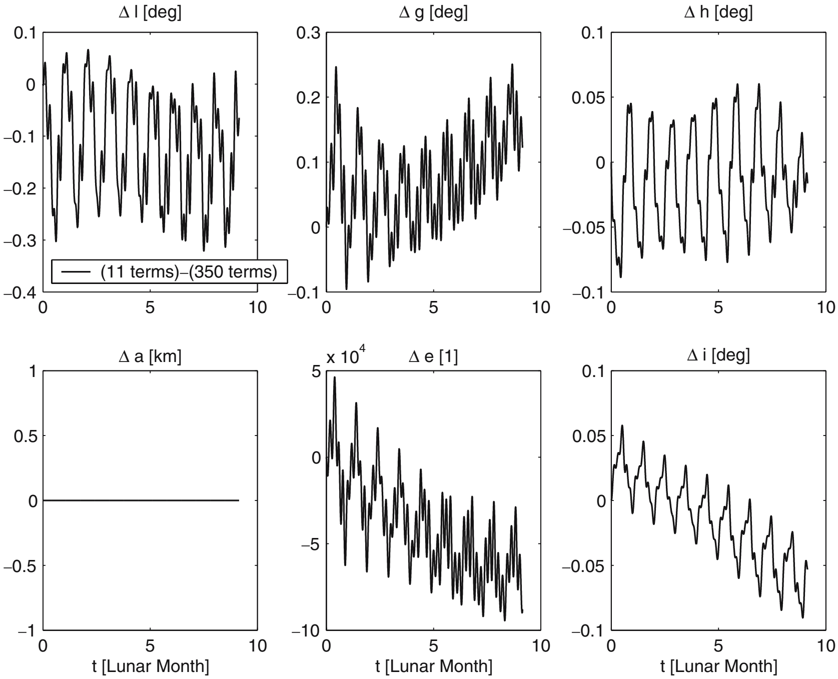
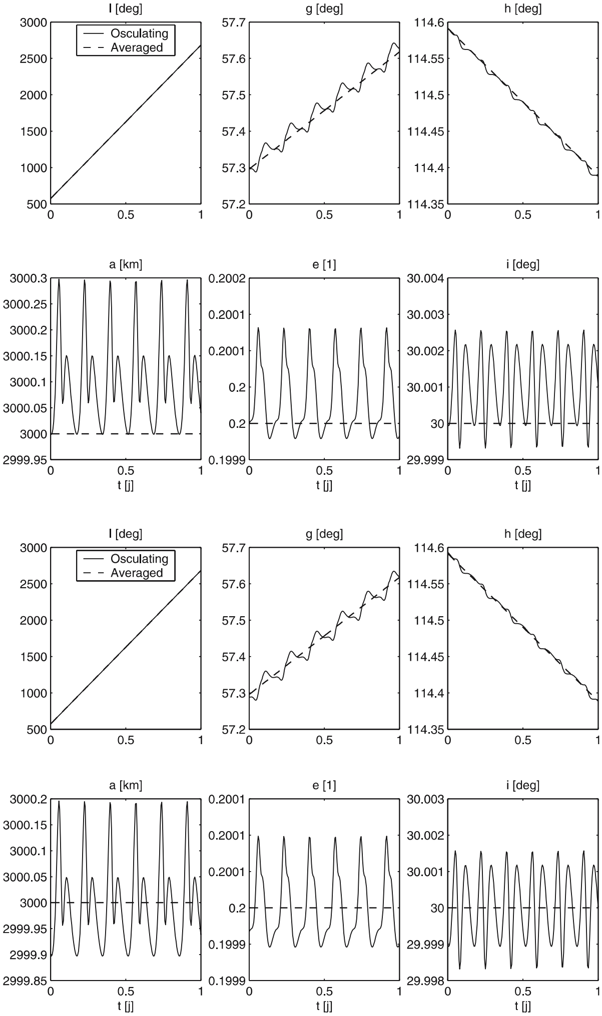


Fig. 6 The effect of the number of terms taken for the Earth: integration of the averaged model (15) for the first-order effects, with ($\epsilon \neq 0, \delta \neq 0, n_{☾} \neq 0, \gamma \neq 0$) and the initial conditions (17). The terrestrial perturbation contained (1) once 11 terms (accuracy 10^{-6}) and (2) once 350 terms (accuracy 10^{-9}); we then plot the difference (1)–(2) in each of the elements, which is of order 10^{-3}

averaged value seems not correct with respect to the osculating one (see Fig. 7, top). In our particular choice, the $\bar{a}$ corresponds to a minimum rather than to the mean value; it should not be the case of course. Note that we can clearly see the period of the satellite on the osculating motion, which is about 4.1 hour for $a = 3,000$ km.

The solution is to adapt the initial conditions, using the same transformation which has been used for averaging the Hamiltonian. One has to be careful that the result is inverted: if we use the direct algorithm of the Lie triangle, then we will have the function $\bar{a} = a + \cdots$ If we rather need the function $a = \bar{a} + \cdots$, then we will have to use the algorithm of the inverse (Henrard 1973) as soon as the second-order is considered. A detailed example of such a transformation is given in Appendix, where we used $\beta = (1 - \eta)/e = e/(1 + \eta)$. If we adapt the initial conditions (l, g, h, a, e, i) in such a way, the correspondence between the averaged value and the osculating is then correct (at the order considered for the transformation), as can be seen in Fig. 7 (bottom).

The goal of our work being to build an averaged theory for mission analysis purposes, transformations from averaged to osculating quantities may be very useful in that context; moreover, this transformation is necessary in order to validate numerically the averaged theory. The transformation can be done exactly since the Lie generators are available and since not any real tracking data of lunar satellite are considered.



$$\left\{\begin{array}{lllll} \bar{l}_0 & = & l_0 & = & 10 \text{ rad} \\ \bar{g}_0 & = & g_0 & = & 1 \text{ rad} \\ \bar{h}_0 & = & h_0 & = & 2 \text{ rad} \\ \bar{a}_0 & = & a_0 & = & 3000 \text{ km} \\ \bar{e}_0 & = & e_0 & = & 0.2 \\ \bar{i}_0 & = & i_0 & = & 30 \text{ deg} \end{array}\right.$$

$$\left\{\begin{array}{lll} \bar{l}_0 & = & 0.10000000 \times 10^2 \text{ rad} \\ \bar{g}_0 & = & 1.00000000 \times 10^0 \text{ rad} \\ \bar{h}_0 & = & 0.20000000 \times 10^1 \text{ rad} \\ \bar{a}_0 & = & 0.30000000 \times 10^4 \text{ km} \\ \bar{e}_0 & = & 0.20000000 \times 10^0 \\ \bar{i}_0 & = & 0.30000000 \times 10^2 \text{ deg} \end{array}\right.$$

$$\left\{\begin{array}{lll} l_0 & = & 0.10000092 \times 10^2 \text{ rad} \\ g_0 & = & 0.99986288 \times 10^0 \text{ rad} \\ h_0 & = & 0.20000347 \times 10^1 \text{ rad} \\ a_0 & = & 0.29998983 \times 10^4 \text{ km} \\ e_0 & = & 0.19996697 \times 10^0 \\ i_0 & = & 0.29999000 \times 10^2 \text{ deg} \end{array}\right.$$

Fig. 7 Comparison of averaged motion (*dashed line*, integration of (15)) with osculating motion (*solid line*, integration of (7)), for the first-order effect of J_2 alone ($\epsilon \neq 0, \delta = 0 = n_{☾} = \gamma$): without adapting the initial conditions (*top*); with adapted initial conditions (*bottom*)

9 Conclusions

We have shown how we built an analytical theory of an artificial satellite of the Moon, by eliminating the short-period terms. We gave some explicit series of the problem, obtained by our home-made algebraic Manipulator software. We performed numerical integrations in order to validate our analytical theory. The effect of each perturbation has been presented progressively and separately as far as possible, in order to achieve a better understanding of the underlying mechanisms. As we could expect, the effect of the Earth plays a major influence, pumping up the eccentricity of the lunar satellite; its role can be modelled already by a few terms only. We stressed and illustrated the importance of adapting the initial conditions from averaged to osculating values in the frame of using an averaged model for mission analysis purposes.

Although not presented in this particular paper, we made other extensive checks (averaged Hamiltonians, generators, liftetimes) with several published works; the results are in good agreement. The results presented here capture the first-order effects of the perturbations by averaging up to order 2, but a full closed-form second-order theory (averaging up to order 4) is also currently being developed and is intended to be published in a forthcoming paper. The third-order will contain the combinations of perturbation parameters ($\epsilon n_☾, \delta n_☾, \gamma n_☾$) and the fourth-order ($\epsilon^2, \delta^2, \gamma^2, \epsilon\delta, \epsilon\gamma, \delta\gamma, \epsilon n_☾^2, \delta n_☾^2, \gamma n_☾^2$).

It is intended to make a more quantitative analysis of the quality of the averaging process, by the same kind of accuracy test as in Knežević and Milani (1995).

Acknowledgements The author is pleased to acknowledge professor Jacques Henrard for fruitful discussions, and for having extended the powerful tool MM ("Moon's series Manipulator") designed by Michèle Moons. A special thank also to Nicolas Rambaux for his help on the numerical integration side.

Appendix: The series giving $a = \bar{a} + \cdots$

Table 3 The *a* series for (96 terms)

	$\bar{f}$	$\bar{g}$	$\bar{h}$	L^*	$\bar{u}$	$\bar{\xi}$	$\bar{a}$	$\bar{n}$	$\bar{e}$	$\bar{\eta}$	$\bar{c}$	$\bar{s}$	δ	ϵ	γ	$1+\bar{c}$	$\bar{\beta}$	$1-\bar{c}$	Coefficient
cos	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0.10000D+01
cos	0	0	0	0	0	0	1	−2	0	0	0	0	0	0	1	0	0	0	−0.24932D+00
cos	0	0	0	0	0	0	1	−2	0	0	2	0	0	0	1	0	0	0	0.74795D+00
cos	0	0	0	0	0	0	1	−2	2	0	0	0	0	0	1	0	0	0	−0.37399D+00
cos	0	0	0	0	0	0	1	−2	2	0	2	0	0	0	1	0	0	0	0.11219D+01
cos	0	0	0	0	0	1	1	−2	0	0	0	0	0	0	1	0	0	0	0.24932D+00
cos	0	0	0	0	0	1	1	−2	0	0	2	0	0	0	1	0	0	0	−0.74795D+00
cos	0	0	0	0	0	1	1	−2	2	0	0	0	0	0	1	0	0	0	0.37399D+00
cos	0	0	0	0	0	1	1	−2	2	0	2	0	0	0	1	0	0	0	−0.11219D+01
cos	0	0	0	0	1	1	1	−2	1	0	0	0	0	0	1	0	0	0	−0.74797D+00
cos	0	0	0	0	1	1	1	−2	1	0	2	0	0	0	1	0	0	0	0.22439D+01
cos	0	0	0	0	1	1	1	−2	3	0	0	0	0	0	1	0	0	0	−0.18699D+00
cos	0	0	0	0	1	1	1	−2	3	0	2	0	0	0	1	0	0	0	0.56096D+00
cos	0	0	0	0	2	1	1	−2	2	0	0	0	0	0	1	0	0	0	0.37399D+00
cos	0	0	0	0	2	1	1	−2	2	0	2	0	0	0	1	0	0	0	−0.11219D+01
cos	0	0	0	0	3	1	1	−2	3	0	0	0	0	0	1	0	0	0	−0.62331D−01
cos	0	0	0	0	3	1	1	−2	3	0	2	0	0	0	1	0	0	0	0.18699D+00
cos	0	0	2	−2	0	0	1	−2	0	0	0	2	0	0	1	0	0	0	0.74450D+00
cos	0	0	2	−2	0	0	1	−2	2	0	0	2	0	0	1	0	0	0	0.11167D+01
cos	0	0	2	−2	0	1	1	−2	0	0	0	2	0	0	1	0	0	0	−0.74450D+00
cos	0	0	2	−2	0	1	1	−2	2	0	0	2	0	0	1	0	0	0	−0.11167D+01
cos	0	0	2	−2	1	1	1	−2	1	0	0	2	0	0	1	0	0	0	0.11167D+01
cos	0	0	2	−2	1	1	1	−2	3	0	0	2	0	0	1	0	0	0	0.27919D+00
cos	0	0	2	−2	2	1	1	−2	2	0	0	2	0	0	1	0	0	0	−0.55837D+00
cos	0	0	2	−2	3	1	1	−2	3	0	0	2	0	0	1	0	0	0	0.93062D−01
cos	0	0	2	−2	−1	1	1	−2	1	0	0	2	0	0	1	0	0	0	0.11167D+01
cos	0	0	2	−2	−1	1	1	−2	3	0	0	2	0	0	1	0	0	0	0.27919D+00
cos	0	0	2	−2	−2	1	1	−2	2	0	0	2	0	0	1	0	0	0	−0.55837D+00
cos	0	0	2	−2	−3	1	1	−2	3	0	0	2	0	0	1	0	0	0	0.93062D−01
cos	0	2	0	0	0	0	1	−2	2	0	0	2	0	0	1	0	0	0	0.18699D+01

Table 3 continued

	$\bar{f}$	$\bar{g}$	$\bar{h}$	L^*	$\bar{u}$	$\bar{\xi}$	$\bar{a}$	$\bar{n}$	$\bar{e}$	$\bar{\eta}$	$\bar{c}$	$\bar{s}$	δ	ϵ	γ	$1+\bar{c}$	$\bar{\beta}$	$1-\bar{c}$	Coefficient
cos	0	2	0	0	0	1	1	−2	2	0	0	2	0	0	1	0	0	0	−0.18699D + 01
cos	0	2	0	0	1	1	1	−2	2	0	0	2	0	0	1	0	−1	0	0.93493D + 00
cos	0	2	0	0	1	1	1	−2	3	0	0	2	0	0	1	0	0	0	0.46747D + 00
cos	0	2	0	0	2	1	1	−2	1	0	0	2	0	0	1	0	−1	0	−0.37397D + 00
cos	0	2	0	0	2	1	1	−2	2	0	0	2	0	0	1	0	0	0	−0.18699D + 00
cos	0	2	0	0	2	1	1	−2	2	1	0	2	0	0	1	0	0	0	−0.37397D + 00
cos	0	2	0	0	3	1	1	−2	2	0	0	2	0	0	1	0	−1	0	0.18699D + 00
cos	0	2	0	0	3	1	1	−2	3	0	0	2	0	0	1	0	0	0	−0.93493D − 01
cos	0	2	0	0	−1	1	1	−2	2	0	0	2	0	0	1	0	1	0	0.93493D + 00
cos	0	2	0	0	−1	1	1	−2	3	0	0	2	0	0	1	0	0	0	0.46747D + 00
cos	0	2	0	0	−2	1	1	−2	1	0	0	2	0	0	1	0	1	0	−0.37397D + 00
cos	0	2	0	0	−2	1	1	−2	2	0	0	2	0	0	1	0	0	0	−0.18699D + 00
cos	0	2	0	0	−2	1	1	−2	2	1	0	2	0	0	1	0	0	0	0.37397D + 00
cos	0	2	0	0	−3	1	1	−2	2	0	0	2	0	0	1	0	1	0	0.18699D + 00
cos	0	2	0	0	−3	1	1	−2	3	0	0	2	0	0	1	0	0	0	−0.93493D − 01
cos	0	2	2	−2	0	0	1	−2	2	0	0	0	0	0	1	2	0	0	0.93062D + 00
cos	0	2	2	−2	0	1	1	−2	2	0	0	0	0	0	1	2	0	0	−0.93062D + 00
cos	0	2	2	−2	1	1	1	−2	2	0	0	0	0	0	1	2	−1	0	0.46531D + 00
cos	0	2	2	−2	1	1	1	−2	3	0	0	0	0	0	1	2	0	0	0.23266D + 00
cos	0	2	2	−2	2	1	1	−2	1	0	0	0	0	0	1	2	−1	0	−0.18612D + 00
cos	0	2	2	−2	2	1	1	−2	2	0	0	0	0	0	1	2	0	0	−0.93062D − 01
cos	0	2	2	−2	2	1	1	−2	2	1	0	0	0	0	1	2	0	0	−0.18612D + 00
cos	0	2	2	−2	3	1	1	−2	2	0	0	0	0	0	1	2	−1	0	0.93062D − 01
cos	0	2	2	−2	3	1	1	−2	3	0	0	0	0	0	1	2	0	0	−0.46531D − 01
cos	0	2	2	−2	−1	1	1	−2	2	0	0	0	0	0	1	2	1	0	0.46531D + 00
cos	0	2	2	−2	−1	1	1	−2	3	0	0	0	0	0	1	2	0	0	0.23266D + 00
cos	0	2	2	−2	−2	1	1	−2	1	0	0	0	0	0	1	2	1	0	−0.18612D + 00
cos	0	2	2	−2	−2	1	1	−2	2	0	0	0	0	0	1	2	0	0	−0.93062D − 01
cos	0	2	2	−2	−2	1	1	−2	2	1	0	0	0	0	1	2	0	0	0.18612D + 00
cos	0	2	2	−2	−3	1	1	−2	2	0	0	0	0	0	1	2	1	0	0.93062D − 01
cos	0	2	2	−2	−3	1	1	−2	3	0	0	0	0	0	1	2	0	0	−0.46531D − 01
cos	0	2	−2	2	0	0	1	−2	2	0	0	0	0	0	1	0	0	2	0.93062D + 00
cos	0	2	−2	2	0	1	1	−2	2	0	0	0	0	0	1	0	0	2	−0.93062D + 00
cos	0	2	−2	2	1	1	1	−2	2	0	0	0	0	0	1	0	−1	2	0.46531D + 00
cos	0	2	−2	2	1	1	1	−2	3	0	0	0	0	0	1	0	0	2	0.23266D + 00
cos	0	2	−2	2	2	1	1	−2	1	0	0	0	0	0	1	0	−1	2	−0.18612D + 00
cos	0	2	−2	2	2	1	1	−2	2	0	0	0	0	0	1	0	0	2	−0.93062D − 01
cos	0	2	−2	2	2	1	1	−2	2	1	0	0	0	0	1	0	0	2	−0.18612D + 00
cos	0	2	−2	2	3	1	1	−2	2	0	0	0	0	0	1	0	−1	2	0.93062D − 01
cos	0	2	−2	2	3	1	1	−2	3	0	0	0	0	0	1	0	0	2	−0.46531D − 01
cos	0	2	−2	2	−1	1	1	−2	2	0	0	0	0	0	1	0	1	2	0.46531D + 00
cos	0	2	−2	2	−1	1	1	−2	3	0	0	0	0	0	1	0	0	2	0.23266D + 00
cos	0	2	−2	2	−2	1	1	−2	1	0	0	0	0	0	1	0	1	2	−0.18612D + 00
cos	0	2	−2	2	−2	1	1	−2	2	0	0	0	0	0	1	0	0	2	−0.93062D − 01
cos	0	2	−2	2	−2	1	1	−2	2	1	0	0	0	0	1	0	0	2	0.18612D + 00
cos	0	2	−2	2	−3	1	1	−2	2	0	0	0	0	0	1	0	1	2	0.93062D − 01
cos	0	2	−2	2	−3	1	1	−2	3	0	0	0	0	0	1	0	0	2	−0.46531D − 01
cos	0	0	0	0	0	0	−1	0	0	−3	0	0	0	1	0	0	0	0	0.50000D + 00
cos	0	0	0	0	0	0	−1	0	0	−3	2	0	0	1	0	0	0	0	−0.15000D + 01
cos	0	0	0	0	0	2	−1	0	0	−2	0	0	0	1	0	0	0	0	−0.50000D + 00
cos	0	0	0	0	0	2	−1	0	0	−2	2	0	0	1	0	0	0	0	0.15000D + 01
cos	1	0	0	0	0	2	−1	0	1	−2	0	0	0	1	0	0	0	0	−0.50000D + 00
cos	1	0	0	0	0	2	−1	0	1	−2	2	0	0	1	0	0	0	0	0.15000D + 01
cos	1	2	0	0	0	2	−1	0	1	−2	0	2	0	1	0	0	0	0	0.75000D + 00
cos	2	2	0	0	0	2	−1	0	0	−2	0	2	0	1	0	0	0	0	0.15000D + 01
cos	3	2	0	0	0	2	−1	0	1	−2	0	2	0	1	0	0	0	0	0.75000D + 00

Table 3 continued

	$\bar{f}$	$\bar{g}$	$\bar{h}$	L^*	$\bar{u}$	$\bar{\xi}$	$\bar{a}$	$\bar{n}$	$\bar{e}$	$\bar{\eta}$	$\bar{c}$	$\bar{s}$	δ	ϵ	γ	$1+\bar{c}$	$\bar{\beta}$	$1-\bar{c}$	Coefficient
cos	0	0	2	0	0	0	−1	0	0	−3	0	2	1	0	0	0	0	0	0.30000D + 01
cos	0	0	2	0	0	2	−1	0	0	−2	0	2	1	0	0	0	0	0	−0.30000D + 01
cos	1	0	2	0	0	2	−1	0	1	−2	0	2	1	0	0	0	0	0	−0.15000D + 01
cos	1	0	−2	0	0	2	−1	0	1	−2	0	2	1	0	0	0	0	0	−0.15000D + 01
cos	1	2	2	0	0	2	−1	0	1	−2	0	0	1	0	0	2	0	0	−0.75000D + 00
cos	1	2	−2	0	0	2	−1	0	1	−2	0	0	1	0	0	0	0	2	−0.75000D + 00
cos	2	2	2	0	0	2	−1	0	0	−2	0	0	1	0	0	2	0	0	−0.15000D + 01
cos	2	2	−2	0	0	2	−1	0	0	−2	0	0	1	0	0	0	0	2	−0.15000D + 01
cos	3	2	2	0	0	2	−1	0	1	−2	0	0	1	0	0	2	0	0	−0.75000D + 00
cos	3	2	−2	0	0	2	−1	0	1	−2	0	0	1	0	0	0	0	2	−0.75000D + 00

References

Brouwer, D.: Solution of the problem of artificial satellite theory without air drag. Astron. J. **64**, 378–397 (1959)

Chapront-Touzé, M., Chapront, J.: Lunar Tables and Programs 4000 BC to AD 8000. Willmann- Bell. (1991)

De Saedeleer, B., Henrard, J.: Orbit of a lunar artificial satellite: analytical theory of perturbations. *IAU Colloq. 196: Transits of Venus: New Views of the Solar System and Galaxy.* pp. 254–262 (2005)

De Saedeleer, B., Henrard, J.: The combined effect of J_2 and C_{22} on the critical inclination of a lunar orbiter'. Adv. Space Res. **37**(1): 80–87 (2006). The Moon and Near-Earth Objects. Also available as http://dx.doi.org/10.1016/j.asr.2005.06.052

De Saedeleer, B.: Analytical theory of an artificial satellite of the Moon. In: Belbruno, E., Gurfil, P. (eds.) Astrodynamics, Space Missions, and Chaos, of the Annals of the New York Academy of Sciences. *Proceedings of the Conference New Trends in Astrodynamics and Applications, January 20-22, 2003*, Vol. 1017, pp. 434–449. Washington (2004)

De Saedeleer, B.: Complete zonal problem of the artificial satellite: generic compact analytic first order in closed form. Celest. Mech. Dynam. Astron. **91**, 239–268 (2005)

Deprit, A.: Canonical transformations depending on a small parameter. Celest. Mech. **1**, 12–30 (1969)

Henrard, J.: The algorithm of the inverse for Lie transform. In: Szebehely, V., Tapley, B. (eds.) *ASSL: Recent Advances in Dynamical Astronomy*, Vol. 39, pp. 248–257. Dordrecht (1973)

Jefferys, W.: Automated, closed form integration of formulas in elliptic motion. Celest. Mech. **3**, 390–394 (1971)

Jupp, A.: The critical inclination problem: 30 years of progress. Celest. Mech. **43**, 127–138 (1988)

Knežević, Z., Milani, A.: Perturbation theory for low satellites: an application. Bull. Astron. Belgrade **152**, 35–48 (1995)

Knežević, Z., Milani, A.: Orbit maintenance of a lunar polar orbiter. Planet. Space Sci. **46**, 1605–1611 (1998)

Konopliv, A. S., Sjogren, W.L., Wimberly, R.N., Cook, R.A., Vijayaraghavan, A.: A high resolution lunar gravity field and predicted orbit behavior. In *Advances in the Astronautical Sciences, Astrodynamics (AAS/AIAA Astrodynamics Specialist Conference, Pap. # AAS 93–622, Victoria, B.C.)*, Vol. 85, pp. 1275–1295 (1993)

Kozai, Y.: Second-order solution of artificial satellite theory without air drag. Astron. J. **67**, 446–461 (1962)

Liu, L., Wang, X.: On the orbital lifetime of high-altitude satellites. Chinese Astron, Astrophys. **24**, 284–288 (2000)

Milani, A., Knežević, Z.: *Selenocentric proper elements: A tool for lunar satellite mission analysis.* Final Report of a study conducted for ESA, ESTEC, Noordwijk (1995)

Oesterwinter, C.: The motion of a lunar satellite. Celest. Mech. **1**, 368–436 (1970)

Press, W. H., Teukolsky, S. A., Vetterling, W. T., Flannery, B. P.: Numerical Recipes in Fortran 77—The Art of Scientific Computing. Cambridge University Press, Cambridge (1986)

Roy, A.: The theory of the motion of an artificial lunar satellite I. Development of the disturbing function. Icarus **9**, 82–132 (1968)

Shniad, H.: The equivalence of von Zeipel mappings and Lie transforms. Celest. Mech. **2**, 114–120 (1970)
Steichen, D.: An averaging method to study the motion of lunar artificial satellites—I: Disturbing function. Celest. Mech. Dynam. Astron. **68**, 205–224 (1998a)
Steichen, D.: An averaging method to study the motion of lunar artificial satellites—II: Averaging and applications. Celest. Mech. Dynam. Astron. **68**, 225–247 (1998b)
Szebehely, V.: Adventures in Celestial Mechanics. University of Texas Press, Texas (1989)