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# COMPLETE ZONAL PROBLEM OF THE ARTIFICIAL SATELLITE: GENERIC COMPACT ANALYTIC FIRST ORDER IN CLOSED FORM

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**Abstract.** This paper is a contribution to the Theory of the Artificial Satellite, within the frame of the Lie Transform as canonical perturbation technique (elimination of the short period terms). We consider the perturbation by any zonal harmonic $J_n$ ($n \geq 2$) of the primary on the satellite, what we call here the *complete zonal problem of the artificial satellite*. This is quite useful for primaries with symmetry of revolution. We give an analytical formula to compute directly the first order averaged Hamiltonian. The computation is carried out in closed form for all terms, avoiding therefore tedious expansions in the eccentricity or in any anomaly; this feature makes the averaging process, not only valid for all kind of elliptic trajectories but at the same time it yields the averaged Hamiltonian in a very short and compact way. The formula allows us to now skip the averaging process, which means an asymptotic gain of a factor $3n/2$ regarding the computational cost of the $n^{\text{th}}$ zonal. Our analytical formulae have been widely checked, by comparison on one hand with published works (Brouwer, 1959) (which contained results for particular zonal harmonics, let's say typically from $J_2$ to $J_8$), and on the other hand with the results of 3 symbolic manipulation software, among which the MM (standing for 'Moon's series Manipulator'), which has already been used and described in (De Saedeleer B., 2004). Additionally, the first order generator associated with this transformation is given into the same closed form, and has also been validated.

**Key words:** artificial satellite theory, Lie, Hamiltonian, zonal harmonic, first order, closed form

## 1. Introduction

The main problem of the artificial satellite has drawn much attention in the past; but the problem was by definition limited to the effect of $J_2$, even if pushed to higher orders. At the opposite side, we want to consider here the complete zonal problem of the artificial satellite, which means taking any zonal $J_n$ into account, but computing only the first order averaged Hamiltonian for the time being. This is quite useful since zonal harmonics are suited to describe primaries with symmetry of revolution, which are common and widely used in many analytical theories of an artificial satellite.

Some averaging techniques for perturbed two-body problems have been fundamental in the understanding of averaging theories and Lie transformations for two-body problems in closed form, like among others (Kozai, 1962; Jefferys, 1971; Coffey and Deprit, 1982; Kelly, 1989; Osácar and Palacián, 1994). No doubt then that the integrals appearing in this complete zonal problem could be solved by several methods described in the literature.

But the fact is that the user still has to perform many steps involving many manipulations before having access to the first order averaged Hamiltonian. In this paper, we derived an analytical formula which gives directly the final result for any zonal $J_n$ in a very compact and efficient way; a formula that we did not find as such in the literature, to the best of our knowledge. On the way, we give also analytical expressions for the mean values over $f$ of the functions $\cos^i(f)$, $\cos^i(f)\cos(af)$ and $\cos^i(f)\sin(af)$, formulae that we did also not find as such in the literature.

Of course some expressions have already been derived for the first order averaged zonal effect, but there were always particular to a given zonal harmonic, let's say typically from $J_2$ to $J_8$. These expressions, together with others derived by use of a symbolic manipulation software, allowed us to check our analytical formula.

It is a fact that canonical perturbation methods have many advantages (Boccaletti and Pucacco, 1999). So we are working in the canonical frame suggested by Hamilton; and we use here the algorithm of the Lie Transform by Hori (1966) and Deprit (1969) for averaging the Hamiltonian of the problem, in canonical variables. The solution is developed in powers of the small factor $\epsilon_n$ linked to each $J_n$. The results are obtained in a closed form, without any series developments in eccentricity or inclination.

The structure of this paper is the following. We apply the Lie Transform method to the complete zonal problem of the artificial satellite (first order) in Section 2. We then develop the solution of the first order averaged Hamiltonian $\bar{\mathcal{H}}_{0,n}^{(1)}$ for the case $n$ even and $n$ odd in Sections 3 and 4 respectively, making use of 3 Propositions, given in Appendices A, B, C. We summarize these results in Section 5 valid for any $n$. We then check our analytical result in Section 6, and estimate its efficiency in Section 7. We do essentially the same for the generator $\mathcal{W}_{1,n}$ in Sections 8 to 11. We then conclude in Section 12.

## 2. The Lie Transform Applied to the Complete Zonal Problem of the Artificial Satellite: First Order Averaged Hamiltonian

Let's consider the complete zonal problem of the artificial satellite. We define the inertial frame $(x, y, z)$ as follows: the origin is taken at the center of the primary; the $(x, y)$ plane is the equatorial plane of the primary, and the $z$

direction is the right-handed normal to $(x, y)$. In order to be able to use the expressions of the spherical harmonics for the potential, we first have to define spherical coordinates $(r, \lambda, \phi)$, so that the latitude $\phi$ is defined as the deviation from the $(x, y)$ plane, while the longitude $\lambda$ is useless here.

Within that inertial frame, the Hamiltonian describing the motion of the artificial satellite under the effect of any zonal harmonic is written:

$$\mathcal{H} = \frac{1}{2}v^2 - \frac{\mu}{r} + \frac{\mu}{r}\sum_{n=1}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\phi) \tag{1}$$

$P_n(x)$ being the Legendre Polynomial of order $n$. If we define the perturbation parameter for the $n^{th}$ zonal harmonic by

$$\epsilon_n = J_n R^n \tag{2}$$

then (1) is written as:

$$\mathcal{H} = \mathcal{H}_0^{(0)} + \sum_{n=1}^{\infty} \epsilon_n \mathcal{H}_{1,n}^{(0)} \tag{3}$$

with:

$$\mathcal{H}_0^{(0)} = \frac{1}{2}v^2 - \frac{\mu}{r} \tag{4}$$

$$\mathcal{H}_{1,n}^{(0)} = + \frac{\mu}{r^{n+1}} P_n(\sin\phi) \tag{5}$$

The Hamiltonian (3), which includes the central field term $\mathcal{H}_0^{(0)}$ and the perturbative partial potential $\mathcal{H}_{1,n}^{(0)}$, is now well suited for a perturbation method. We use here the method of the Lie Transform (Deprit, 1969). The Hamiltonian is written in powers of a perturbative parameter $\epsilon_n$. For the initial Hamiltonian (*input*), we write: $\mathcal{H}_n^{(0)} = \sum_{i\geqslant 0} \frac{\epsilon_n^i}{i!}\mathcal{H}_{i,n}^{(0)}$; and for the transformed Hamiltonian (*output*), where the fast angle has been removed, we write: $\mathcal{H}_{0,n} = \sum_{i\geqslant 0} \frac{\epsilon_n^i}{i!}\mathcal{H}_{0,n}^{(i)}$. The Lie algorithm consists of imposing the form of the new Hamiltonian; here we choose it to be independent of the fastest angle $l$:

$$\mathcal{H}_{0,n}^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_{1,n}^{(0)} \mathrm{d}l = \cdots = \bar{\mathcal{H}}_{0,n}^{(1)} \tag{6}$$

We write $\mathcal{H}_{0,n}^{(1)}$ as $\bar{\mathcal{H}}_{0,n}^{(1)}$ in order to remember that $l$ has been eliminated. The generator $\mathcal{W}_{1,n}$ associated with this transformation may be computed by using the relation

$$\mathcal{H}_{0,n}^{(1)} = \mathcal{H}_{1,n}^{(0)} + \left(\mathcal{H}_0^{(0)}; \mathcal{W}_{1,n}\right) \tag{7}$$

$\mathcal{H}_0^{(0)}$ being a function of $L$ only, the Poisson parenthesis reduces to one term, giving:

$$\mathcal{W}_{1,n} = \frac{1}{n^*}\int_0^l \left(\mathcal{H}_{1,n}^{(0)} - \bar{\mathcal{H}}_{0,n}^{(1)}\right) \mathrm{d}l \tag{8}$$

if we define $n^* = \frac{\mu^2}{L^3}$.

Our goal in this paper is to obtain an analytical expression which gives directly $\bar{\mathcal{H}}_{0,n}^{(1)}$ for a given $n$. Doing so, we will be able in the future to skip the averaging process of $\mathcal{H}_{1,n}^{(0)}$ since we know directly the averaged Hamiltonian; this drastically reduces the number of terms being manipulated (see Section 7). A similar analytical expression will also be obtained for the generator $\mathcal{W}_{1,n}$.

We come back on the choice of the variables now. In order to keep the Hamiltonian formalism, it is required to work in canonical variables; we choose the classical Delaunay variables $(q_i, p_i) = (l, g, h, L, G, H)$. Now we have to write the Hamiltonians in these variables. The central term $\frac{1}{2}v^2 - \frac{\mu}{r}$ is simply written $\mathcal{H}_0^{(0)} = -\frac{\mu^2}{2L^2}$. For the partial perturbative Hamiltonian $\mathcal{H}_{1,n}^{(0)}$, there is no problem to translate its argument $(\sin\phi)$ into Delaunay variables: it can be done by way of spherical trigonometry (the plane of the orbit being at an inclination $I$):

$$\sin\phi = \sin I \sin(f + g) \tag{9}$$

But there remains the variable $r$ and $f$ to be expressed as a function of $(l, g, h)$ in order to be able to apply a canonical perturbation method. It turns out that the functions $r = r(l, g, h)$ and $f = f(l, g, h)$ cannot be expressed in a closed form; so we prefer to use the following set of auxiliary variables $(\xi, f, g, h, a, n^*, e, \eta, s, c)$, which is closed and allows high eccentricities:

$$\begin{array}{ll}
\xi = \frac{a}{r} = \frac{1+e\cos f}{1-e^2} = \frac{1}{1-e\cos E} & f \\
a = \frac{L^2}{\mu} & n^* = \frac{\mu^2}{L^3} \\
e = \sqrt{1 - \left(\frac{G}{L}\right)^2} & \eta = \sqrt{1-e^2} = \frac{G}{L} \\
s = \sin I = \sqrt{1 - \left(\frac{H}{G}\right)^2} & c = \cos I = \frac{H}{G} \\
g & h
\end{array} \tag{10}$$

Note that we used the notation $n^*$ in order not to be confused with the index $n$ already introduced in (1). The only drawback of this set is that it is redundant and that we would need to perform partial derivatives of them with respect to the canonical variables in case we have to compute a Poisson bracket, which is by the way not the case here for the first order averaged Hamiltonian (but it would be anyway not too burdensome). The only partial derivative that we will need is

$$\frac{\partial f}{\partial l} = \xi^2 \eta \tag{11}$$

which plays an important role, since it will allow us to switch the integration from $l$ to $f$:

$$\mathcal{H}_{0,n}^{(1)} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}_{1,n}^{(0)}\,\mathrm{d}l = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathcal{H}_{1,n}^{(0)}}{\xi^2\eta}\,\mathrm{d}f = \frac{1}{2\pi}\int_0^{2\pi} \Delta\,\mathrm{d}f = \bar{\mathcal{H}}_{0,n}^{(1)} \tag{12}$$

In the following, we will denote the quantity $\frac{\mathcal{H}_{1,n}^{(0)}}{\xi^2\eta}$ by $\Delta$. So our aim is now to recognize in $\Delta$ the part which is independent of $f$, in order to obtain directly $\bar{\mathcal{H}}_{0,n}^{(1)}$.

We start by expressing (5) in the new set of variables (10):

$$\mathcal{H}_{1,n}^{(0)} = +\frac{\mu}{r^{n+1}} P_n(\sin\phi) = n^{*2}a^3\left(\frac{\xi}{a}\right)^{n+1} P_n(\sin\phi) = n^{*2}\frac{\xi^{n+1}}{a^{n-2}} P_n(\sin\phi) \tag{13}$$

and furthermore we convert it into $\Delta$ for the integration in $f$:

$$\Delta = n^{*2}\frac{\xi^{n-1}}{\eta a^{n-2}} P_n(\sin\phi) \tag{14}$$

The integration will be possible since $\xi$ will appear only at the numerator. Note that the variable $h$ does not appear anywhere, because of the symmetry of revolution of the zonal problems.

Even if we search for the most compact analytical result, there is no way to avoid the expansion of some functions; this it the price to pay to extract the first order mean value. Among these function is the Legendre polynomial $P_n(x)$ which can be expressed by the following expansion (see **8.911** 1. of (Gradshteyn and Ryzhik, 1980)):

$$P_n(x) = \frac{1}{2^n}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k} \tag{15}$$

where the lower integer part of $n$ is denoted by $\lfloor n \rfloor$.

In (15) the argument $x$ will be $\sin\phi = s\sin(f+g)$ by (9), so that we have to consider $\sin^{n-2k}(f+g)$. The $\sin(x)$ function has itself the following expansions (see **1.320** 1. of (Gradshteyn and Ryzhik, 1980)):

$$\sin^{2n}(x) = \frac{1}{2^{2n}}\left\{\sum_{k=0}^{n-1}(-1)^{n-k}2\binom{2n}{k}\cos 2(n-k)x + \binom{2n}{n}\right\} \tag{16}$$

$$\sin^{2n-1}(x) = \frac{1}{2^{2n-2}}\sum_{k=0}^{n-1}(-1)^{n+k-1}\binom{2n-1}{k}\sin(2n-2k-1)x \tag{17}$$

where $\binom{n}{k}$ stands for $\frac{n!}{k!(n-k)!}$.

## 3. Analytical Expression of $\bar{\mathcal{H}}^{(1)}_{0,n}$ for the $n$ Even Case

To go further in the developments, we assume starting from now $n$ to be *even* (and $n \geqslant 2$), so that (16) may be rewritten for an even $n$:

$$\sin^n(x) = \frac{1}{2^n}\left\{\sum_{k=0}^{\frac{n}{2}-1}(-1)^{\frac{n}{2}-k}2\binom{n}{k}\cos(n-2k)x + \binom{n}{\frac{n}{2}}\right\} \tag{18}$$

and furthermore for the even power $n-2k$ appearing in (15):

$$\sin^{n-2k}(x) = \frac{1}{2^{n-2k}}\left\{\sum_{p=0}^{\frac{n}{2}-k-1}(-1)^{\frac{n}{2}-k-p}2\binom{n-2k}{p}\right.$$
$$\left.\times\cos(n-2k-2p)x + \binom{n-2k}{\frac{n}{2}-k}\right\} \tag{19}$$

On the other side, the Newton's formula

$$(x+y)^n = \sum_{i=0}^{n} x^i y^{n-i}\binom{n}{i} \tag{20}$$

may be applied to expand $\xi^{n-1}$:

$$\xi^{n-1} = \left(\frac{1+e\cos f}{\eta^2}\right)^{n-1} = \frac{1}{\eta^{2n-2}}\sum_{i=0}^{n-1}(e\cos f)^i 1^{n-1-i}\binom{n-1}{i} \tag{21}$$

With (21) and (19), (14) may now be rewritten as

$$\Delta = n^{*2}\frac{1}{\eta a^{n-2}}\left\{\frac{1}{\eta^{2n-2}}\sum_{i=0}^{n-1}e^i\cos^i f\binom{n-1}{i}\right\}\frac{1}{2^n}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}\frac{s^{n-2k}}{2^{n-2k}}$$
$$\times\left\{\sum_{p=0}^{\frac{n}{2}-k-1}(-1)^{\frac{n}{2}-k-p}2\binom{n-2k}{p}\cos(n-2k-2p)(f+g) + \binom{n-2k}{\frac{n}{2}-k}\right\} \tag{22}$$

Let's now build the following definitions:

$$\alpha_{i,n}(e) = e^i\binom{n-1}{i}$$
$$\beta_{k,n}(s) = \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}\frac{s^{n-2k}}{2^{n-2k}} \tag{23}$$
$$\gamma_{p,k,n} = (-1)^{\frac{n}{2}-k-p}2\binom{n-2k}{p}$$
$$\delta_n = \frac{n^{*2}}{2^n\eta^{2n-1}a^{n-2}}$$

which allow us to write (22) in a more compact form, followed by further summation manipulations in order to gather the trigonometric functions:

$$\Delta = \delta_n \left\{ \sum_{i=0}^{n-1} \alpha_{i,n}(e) \cos^i f \right\} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \left\{ \sum_{p=0}^{\frac{n}{2}-k-1} \gamma_{p,k,n} \cos(n-2k-2p)(f+g) + \binom{n-2k}{\frac{n}{2}-k} \right\}$$

$$= \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \left\{ \sum_{i=0}^{n-1} \alpha_{i,n}(e) \cos^i f \right\} \left\{ \sum_{p=0}^{\frac{n}{2}-k-1} \gamma_{p,k,n} \cos(n-2k-2p)(f+g) + \binom{n-2k}{\frac{n}{2}-k} \right\}$$

$$= \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \left\{ \sum_{i=0}^{n-1} \sum_{p=0}^{\frac{n}{2}-k-1} \alpha_{i,n}(e)\ \gamma_{p,k,n} \cos^i f \cos(n-2k-2p)(f+g) + \binom{n-2k}{\frac{n}{2}-k} \sum_{i=0}^{n-1} \alpha_{i,n}(e) \cos^i f \right\}$$

$$= \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \{A + B\} \qquad (24)$$

where we have split into

$$A = \sum_{i=0}^{n-1} \sum_{p=0}^{\frac{n}{2}-k-1} \alpha_{i,n}(e)\ \gamma_{p,k,n} \cos^i f \cos(n-2k-2p)(f+g)$$

$$B = \binom{n-2k}{\frac{n}{2}-k} \sum_{i=0}^{n-1} \alpha_{i,n}(e) \cos^i f \qquad (25)$$

so that we may obtain the first order averaged Hamiltonian in that way:

$$\bar{\mathcal{H}}_{0,n}^{(1)} = \frac{1}{2\pi} \int_0^{2\pi} \Delta\, \mathrm{d}f = \frac{\delta_n}{2\pi} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \left\{ \int_0^{2\pi} A\, \mathrm{d}f + \int_0^{2\pi} B\, \mathrm{d}f \right\} \qquad (26)$$

Let's first consider the integration of $B$ over $2\pi$. We make use of the Proposition 1 of the Appendix A:

$$\int_0^{2\pi} B\,\mathrm{d}f = \int_0^{2\pi} \binom{n-2k}{\frac{n}{2}-k} \sum_{i=0}^{n-1} \alpha_{i,n}(e)\cos^i f\,\mathrm{d}f$$
$$= \binom{n-2k}{\frac{n}{2}-k} \sum_{i=0}^{n-1} \alpha_{i,n}(e) \int_0^{2\pi} \cos^i f\,\mathrm{d}f$$
$$= \binom{n-2k}{\frac{n}{2}-k} \sum_{\text{even } i=0}^{n-1} \alpha_{i,n}(e) \frac{1}{2^i} \binom{i}{\frac{i}{2}} 2\pi \tag{27}$$

Let's now integrate $A$:

$$\int_0^{2\pi} A\,\mathrm{d}f = \sum_{i=0}^{n-1} \sum_{p=0}^{\frac{n}{2}-k-1} \alpha_{i,n}(e)\gamma_{p,k,n} \int_0^{2\pi} \cos^i f \cos(n-2k-2p)(f+g)\,\mathrm{d}f \tag{28}$$

We may separate $f$ from $g$ in the integrand by using the formula $\cos(u+v) = \cos u \cos v - \sin u \sin v$:

$$\int_0^{2\pi} \cos^i f \cos(n-2k-2p)(f+g)\,\mathrm{d}f$$
$$= \int_0^{2\pi} \cos^i f \{\cos(n-2k-2p)f\ \cos(n-2k-2p)g$$
$$- \sin(n-2k-2p)f\ \sin(n-2k-2p)g\}\,\mathrm{d}f$$
$$= \cos(n-2k-2p)g \int_0^{2\pi} \cos^i f \cos(n-2k-2p)f\,\mathrm{d}f$$
$$- \sin(n-2k-2p)g \int_0^{2\pi} \cos^i f \sin(n-2k-2p)f\,\mathrm{d}f \tag{29}$$

We apply now the Proposition 3 of the Appendix C with $a = (n-2k-2p)$:

$$\int_0^{2\pi} \cos^i f \sin(n-2k-2p)f\,\mathrm{d}f = 0 \tag{30}$$

and the PROPOSITION 2 of the Appendix B with $a = (n-2k-2p)$:

$$\int_0^{2\pi} \cos^i f \cos(n-2k-2p)f\,\mathrm{d}f = \frac{\pi}{2^{i-1}} \binom{i}{\frac{i-(n-2k-2p)}{2}} \tag{31}$$

if $i$ and $(n-2k-2p)$ are of the same parity; that is to say for $i$ even since $n$ is assumed to be even.

Using (31), (28) becomes:

$$\int_0^{2\pi} A\,\mathrm{d}f = \sum_{\text{even } i=0}^{n-1} \sum_{p=0}^{\frac{n}{2}-k-1} \alpha_{i,n}(e)\ \gamma_{p,k,n} \frac{\pi}{2^{i-1}} \binom{i}{\frac{i-(n-2k-2p)}{2}} \cos(n-2k-2p)g \tag{32}$$

Gathering the results (32) and (27) and canceling the $2\pi$, (26) becomes finally:

$$\begin{aligned}\bar{\mathcal{H}}_{0,n}^{(1)} &= \frac{1}{2\pi}\int_0^{2\pi} \Delta \, \mathrm{d}f = \frac{\delta_n}{2\pi}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \beta_{k,n}(s)\left\{\int_0^{2\pi} A \, \mathrm{d}f + \int_0^{2\pi} B \, \mathrm{d}f\right\} \\ &= \delta_n \sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \beta_{k,n}(s) \sum_{\text{even } i=0}^{n-1} \frac{\alpha_{i,n}(e)}{2^i}\left[\binom{i}{\frac{i}{2}}\binom{n-2k}{\frac{n}{2}-k}\right. \\ &\quad \left. + \sum_{p=0}^{\frac{n}{2}-k-1} \gamma_{p,k,n}\binom{i}{\frac{i-(n-2k-2p)}{2}} \cos(n-2k-2p)g\right] \end{aligned} \tag{33}$$

with the definitions (23).

## 4. Analytical Expression of $\bar{\mathcal{H}}_{0,n}^{(1)}$ for the $n$ Odd Case

One can do the same kind of developments assuming $n$ odd now (and $n \geqslant 3$). The Legendre polynomial $P_n(x)$ is still expanded by (15), and we still have to consider $\sin^{n-2k}(f+g)$, but now the exponent is odd, and instead of (16) we have to use (17), which may be rewritten for an odd $n$:

$$\sin^n(x) = \frac{1}{2^{n-1}}\left\{\sum_{k=0}^{\frac{n-1}{2}} (-1)^{\frac{n-1}{2}+k}\binom{n}{k}\sin(n-2k)x\right\} \tag{34}$$

and furthermore for the odd power $n-2k$ appearing in (15):

$$\sin^{n-2k}(x) = \frac{1}{2^{n-2k-1}}\left\{\sum_{p=0}^{\frac{n-1}{2}-k} (-1)^{\frac{n-1}{2}-k+p}\binom{n-2k}{p}\sin(n-2k-2p)x\right\} \tag{35}$$

Using the same expansion (21) together with (35), (14) may now be rewritten by $\Delta^*$ for $n$ odd as:

$$\begin{aligned}\Delta^* &= \frac{\mathcal{H}_{1,n}^{(0)}}{\xi^2\eta} \\ &= n^{*2}\frac{1}{\eta a^{n-2}}\left\{\frac{1}{\eta^{2n-2}}\sum_{i=0}^{n-1} e^i \cos^i f \binom{n-1}{i}\right\} \\ &\quad \times \frac{1}{2^n}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor} \frac{(-1)^k(2n-2k)!}{k!(n-k)!(n-2k)!}\frac{s^{n-2k}}{2^{n-2k}}\frac{1}{2^{-1}} \\ &\quad \times \left\{\sum_{p=0}^{\frac{n-1}{2}-k} (-1)^{\frac{n-1}{2}-k+p}\binom{n-2k}{p}\sin(n-2k-2p)(f+g)\right\} \end{aligned} \tag{36}$$

If we take again the definitions (23) with another one:

$$\zeta_{p,k,n} = (-1)^{\frac{n-1}{2}-k+p}\binom{n-2k}{p} \tag{37}$$

one can write (36) in a more compact form. Moreover, one can unify the definition of the symbols for the odd and the even case, by defining:

$$\gamma^*_{p,k,n} = (-1)^{\lfloor\frac{n}{2}\rfloor-k-p}\; 2\binom{n-2k}{p} = \begin{cases} \gamma_{p,k,n} & \text{for } n \text{ even,} \\ 2\zeta_{p,k,n} & \text{for } n \text{ odd.} \end{cases} \tag{38}$$

So that (36) may be rewritten and further manipulated in order to gather the trigonometric functions:

$$\begin{aligned} \Delta^* &= \delta_n\left\{\sum_{i=0}^{n-1}\alpha_{i,n}(e)\cos^i f\right\}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}2\beta_{k,n}(s)\left\{\sum_{p=0}^{\frac{n-1}{2}-k}\frac{\gamma^*_{p,k,n}}{2}\sin(n-2k-2p)(f+g)\right\} \\ &= \delta_n\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\beta_{k,n}(s)\left\{\sum_{i=0}^{n-1}\alpha_{i,n}(e)\cos^i f\right\}\left\{\sum_{p=0}^{\frac{n-1}{2}-k}\gamma^*_{p,k,n}\sin(n-2k-2p)(f+g)\right\} \\ &= \delta_n\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\beta_{k,n}(s)\left\{\sum_{i=0}^{n-1}\sum_{p=0}^{\frac{n-1}{2}-k}\alpha_{i,n}(e)\;\gamma^*_{p,k,n}\cos^i f\sin(n-2k-2p)(f+g)\right\} \\ &= \delta_n\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\beta_{k,n}(s)\{A^*\} \end{aligned} \tag{39}$$

where we have defined $A^*$ as:

$$A^* = \sum_{i=0}^{n-1}\sum_{p=0}^{\frac{n-1}{2}-k}\alpha_{i,n}(e)\;\gamma^*_{p,k,n}\cos^i f\sin(n-2k-2p)(f+g) \tag{40}$$

so that we will obtain the first order averaged Hamiltonian by

$$\bar{\mathcal{H}}^{(1)}_{0,n} = \frac{1}{2\pi}\int_0^{2\pi}\Delta^*\;\mathrm{d}f = \frac{\delta_n}{2\pi}\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\beta_{k,n}(s)\left\{\int_0^{2\pi}A^*\;\mathrm{d}f\right\} \tag{41}$$

Note that there is no $B^*$ for the $n$ odd case. Let's now integrate $A^*$:

$$\int_0^{2\pi}A^*\;\mathrm{d}f = \sum_{i=0}^{n-1}\sum_{p=0}^{\frac{n-1}{2}-k}\alpha_{i,n}(e)\;\gamma^*_{p,k,n}\int_0^{2\pi}\cos^i f\sin(n-2k-2p)(f+g)\;\mathrm{d}f \tag{42}$$

We may again separate $f$ from $g$ in the integrand by using the formula $\sin(u+v) = \sin u\cos v + \cos u\sin v$:

$$\begin{aligned}
&\int_0^{2\pi} \cos^i f \sin(n-2k-2p)(f+g)\ \mathrm{d}f \\
&= \int_0^{2\pi} \cos^i f \Big\{ \sin(n-2k-2p)f\ \cos(n-2k-2p)g \\
&\quad + \cos(n-2k-2p)f\ \sin(n-2k-2p)g \Big\}\ \mathrm{d}f \\
&= \cos(n-2k-2p)g \int_0^{2\pi} \cos^i f \sin(n-2k-2p)f\ \mathrm{d}f \\
&\quad + \sin(n-2k-2p)g \int_0^{2\pi} \cos^i f \cos(n-2k-2p)f\ \mathrm{d}f
\end{aligned} \tag{43}$$

We apply now the Proposition 3 of the Appendix C with $a = (n-2k-2p)$:

$$\int_0^{2\pi} \cos^i f \sin(n-2k-2p)f\ \mathrm{d}f = 0 \tag{44}$$

and the Proposition 2 of the Appendix B with $a = (n-2k-2p)$:

$$\int_0^{2\pi} \cos^i f \cos(n-2k-2p)f\ \mathrm{d}f = \frac{\pi}{2^{i-1}} \binom{i}{\frac{i-(n-2k-2p)}{2}} \tag{45}$$

if $i$ and $(n-2k-2p)$ are of the same parity; that is to say for $i$ odd since $n$ is assumed to be odd.

Using (45), (42) becomes:

$$\int_0^{2\pi} A^*\ \mathrm{d}f = \sum_{\text{odd } i=0}^{n-1} \sum_{p=0}^{\frac{n-1}{2}-k} \alpha_{i,n}(e)\ \gamma^*_{p,k,n} \frac{\pi}{2^{i-1}} \binom{i}{\frac{i-(n-2k-2p)}{2}} \sin(n-2k-2p)g \tag{46}$$

Plugging the result (46) into (41), and canceling the $2\pi$, one has finally:

$$\begin{aligned}
\bar{\mathcal{H}}^{(1)}_{0,n} &= \frac{1}{2\pi} \int_0^{2\pi} \Delta^* \mathrm{d}f = \frac{\delta_n}{2\pi} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \left\{ \int_0^{2\pi} A^*\ \mathrm{d}f \right\} \\
&= \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \sum_{\text{odd } i=0}^{n-1} \frac{\alpha_{i,n}(e)}{2^i} \sum_{p=0}^{\frac{n-1}{2}-k} \gamma^*_{p,k,n} \binom{i}{\frac{i-(n-2k-2p)}{2}} \sin(n-2k-2p)g
\end{aligned} \tag{47}$$

with the definitions (23) and (38).

## 5. Summary of the Analytical Expression Giving $\bar{\mathcal{H}}^{(1)}_{0,n}$ for any $n$

In summary, we have obtained an analytical formula which gives the first order averaged Hamiltonian for any zonal harmonic.

For the case $n$ even, we have:

$$\bar{\mathcal{H}}_{0,n}^{(1)} = \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \sum_{\text{even } i=0}^{n-1} \frac{\alpha_{i,n}(e)}{2^i} \left[ \binom{i}{\frac{i}{2}} \binom{n-2k}{\frac{n}{2}-k} + \sum_{p=0}^{\frac{n}{2}-k-1} \gamma^*_{p,k,n} \binom{i}{\frac{i-(n-2k-2p)}{2}} \cos(n-2k-2p)g \right] \quad (48)$$

while for the case $n$ odd, we have:

$$\bar{\mathcal{H}}_{0,n}^{(1)} = \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \sum_{\text{odd } i=0}^{n-1} \frac{\alpha_{i,n}(e)}{2^i} \sum_{p=0}^{\frac{n-1}{2}-k} \gamma^*_{p,k,n} \binom{i}{\frac{i-(n-2k-2p)}{2}} \sin(n-2k-2p)g \quad (49)$$

both formulae making use of the definitions (23) and (38), which are recalled here:

$$\alpha_{i,n}(e) = e^i \binom{n-1}{i}$$

$$\beta_{k,n}(s) = \frac{(-1)^k (2n-2k)!}{k!(n-k)!(n-2k)!} \frac{s^{n-2k}}{2^{n-2k}}$$

$$\gamma^*_{p,k,n} = (-1)^{\lfloor \frac{n}{2} \rfloor - k - p}\, 2 \binom{n-2k}{p}$$

$$\delta_n = \frac{n^{*2}}{2^n \eta^{2n-1} a^{n-2}} \quad (50)$$

A few remarks on these results:

1. There is of course a notable symmetry between the odd and even case, especially $\sin(jg)$ and $\cos(jg)$ series. Physically, odd harmonics are to be considered only if symmetry of the primary with respect to the equator does not exist.
2. The function $\lfloor \frac{n}{2} \rfloor$ may of course be simplified with $\lfloor \frac{n}{2} \rfloor = \begin{cases} \frac{n}{2} & \text{for } n \text{ even,} \\ \frac{n-1}{2} & \text{for } n \text{ odd.} \end{cases}$
3. The function $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ assumes implicitly (i) $n \geq k$, (ii) $n \geq 0$ and (iii) $k \geq 0$. The (i) and (ii) are satisfied for the formulae (48–50), while (iii) gives the restriction $p \geq \frac{n-i-2k}{2}$ for the case $\binom{i}{\frac{i-(n-2k-2p)}{2}}$ only. So that the summation on $p$ starts giving terms from that value only; we take the more easy common convention $\binom{n}{k} = 0$ for $k < 0$. Note that the ending value on $p$ is also sometimes lower than the starting value, then no terms is

generated. It may complicate a little bit the counting for the number of terms, but we have a satisfactory description of these in Section 7.

4. The analytical result makes more easy the considerations about the d'Alembert characteristic, e.g. for the fastest periodic term in $g$, which is $e^i \cos(n - 2k - 2p)g$, with $p = p_{\min} = \frac{n-i-2k}{2}$, we have $e^i \cos ig$, which means that the d'Alembert characteristic is preserved.

## 6. Checks of the Analytical Expression of $\bar{\mathcal{H}}_{0,n}^{(1)}$ for $J_2$ to $J_7$

Many checks of our analytical results (48–50) were performed, by comparison either with the results from other symbolic manipulation software packages or with published works; we present a few of them here. We used three symbolic manipulation software packages: the *Maple* program (for computing the analytical result, and comparing to the output of the *MM* program), the *Mathematica* program (for integrating symbolically (14) and comparing that result to our analytical result), and the *MM (standing for 'Moon's series Manipulator')*, a specific FORTRAN program, which has been developed at our University. The *MM* program is based upon the idea described in (Henrard, 1986). Within this tool, each expression is given by a series of linear trigonometric functions, with polynomial coefficients; an example is given in the Table I. The *MM* program has been used also for integrating (14), a procedure which was used to check some generic subroutines.

Let's first consider a few checks of (48) for $n$ even.

For $n = 2$, the analytical formula (48) gives $\bar{\mathcal{H}}_{0,2}^{(1)} = \frac{n^{*2}}{4\eta^3}(3s^2 - 2)$ which is the same result as Brouwer's (Brouwer, 1959) first order averaged Hamiltonian for $J_2$ (suppressing the heavy notation):

$$F_1^* = \frac{\mu^4 k_2}{L^3 G^3} A \quad \text{with} \quad A = -\frac{1}{2} + \frac{3}{2}\frac{H^2}{G^2} \qquad \text{((13) from Brouwer)}$$

if we consider that $\frac{F_1^*}{-2k_2} = \bar{\mathcal{H}}_{0,2}^{(1)}$ (since $\epsilon_2 = 2k_2$) and if we make the following substitutions:

$$\frac{H}{G} \to c \quad \frac{\mu^2}{L^3} \to n^* \quad \frac{L^2}{\mu} \to a \quad \frac{L}{G} \to \eta^{-1} \quad \eta^2 \to 1 - e^2 \quad c^2 \to 1 - s^2 \tag{51}$$

For $n = 4$, the analytical formula (48) gives:

$$\bar{\mathcal{H}}_{0,4}^{(1)} = \frac{n^{*2}}{16\eta^7 a^2}\left\{\left(\frac{45}{2}s^2 - \frac{105}{4}s^4\right)e^2 \cos(2g) + \left(9 - 45s^2 + \frac{315}{8}s^4\right)e^2 + 6 - 30s^2 + \frac{105}{4}s^4\right\} \tag{52}$$

which is the same result as Brouwer's (Brouwer, 1959, p. 389) first order averaged Hamiltonian for $J_4$:

TABLE I The $\bar{\mathcal{H}}^{(1)}_{0,6}$ series for $J_6$ (20 terms)

| | $f$ | $g$ | $\xi$ | $a$ | $n^*$ | $e$ | $\eta$ | $c$ | $s$ | $(f-l)$ | Coefficient |
|---|---|---|---|---|---|---|---|---|---|---|---|
| cos | 0 | 0 | 0 | −4 | 2 | 0 | −11 | 0 | 0 | 0 | $-0.31250000000000000 \times 10^{0}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 0 | −11 | 0 | 2 | 0 | $0.32812500000000000 \times 10^{1}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 0 | −11 | 0 | 4 | 0 | $-0.73828125000000000 \times 10^{1}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 0 | −11 | 0 | 6 | 0 | $0.45117187500000000 \times 10^{1}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 2 | −11 | 0 | 0 | 0 | $-0.15625000000000000 \times 10^{1}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 2 | −11 | 0 | 2 | 0 | $0.16406250000000000 \times 10^{2}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 2 | −11 | 0 | 4 | 0 | $-0.36914062500000000 \times 10^{2}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 2 | −11 | 0 | 6 | 0 | $0.22558593750000000 \times 10^{2}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 4 | −11 | 0 | 0 | 0 | $-0.58593750000000000 \times 10^{0}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 4 | −11 | 0 | 2 | 0 | $0.61523437500000000 \times 10^{1}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 4 | −11 | 0 | 4 | 0 | $-0.13842773437500000 \times 10^{2}$ |
| cos | 0 | 0 | 0 | −4 | 2 | 4 | −11 | 0 | 6 | 0 | $0.84594726562500000 \times 10^{1}$ |
| cos | 0 | 2 | 0 | −4 | 2 | 2 | −11 | 0 | 2 | 0 | $-0.82031250000000000 \times 10^{1}$ |
| cos | 0 | 2 | 0 | −4 | 2 | 2 | −11 | 0 | 4 | 0 | $0.24609375000000000 \times 10^{2}$ |
| cos | 0 | 2 | 0 | −4 | 2 | 2 | −11 | 0 | 6 | 0 | $-0.16918945312500000 \times 10^{2}$ |
| cos | 0 | 2 | 0 | −4 | 2 | 4 | −11 | 0 | 2 | 0 | $-0.41015625000000000 \times 10^{1}$ |
| cos | 0 | 2 | 0 | −4 | 2 | 4 | −11 | 0 | 4 | 0 | $0.12304687500000000 \times 10^{2}$ |
| cos | 0 | 2 | 0 | −4 | 2 | 4 | −11 | 0 | 6 | 0 | $-0.84594726562500000 \times 10^{1}$ |
| cos | 0 | 4 | 0 | −4 | 2 | 4 | −11 | 0 | 4 | 0 | $-0.76904296875000000 \times 10^{0}$ |
| cos | 0 | 4 | 0 | −4 | 2 | 4 | −11 | 0 | 6 | 0 | $0.84594726562500000 \times 10^{0}$ |

$$F_1^* = \Delta_4 F_{2s}^* + \Delta_4 F_{2p}^* = \frac{\mu^6 k_4}{L^3 G^7}\left\{\left(\frac{3}{8} - \frac{15}{4}\frac{H^2}{G^2} + \frac{35}{8}\frac{H^4}{G^4}\right)\left(\frac{5}{2} - \frac{3}{2}\frac{G^2}{L^2}\right) + \left(-\frac{5}{6} + \frac{20}{3}\frac{H^2}{G^2} - \frac{35}{6}\frac{H^4}{G^4}\right)\left(\frac{3}{4} - \frac{3}{4}\frac{G^2}{L^2}\right)\cos(2g)\right\} \quad (53)$$

if we consider that $\frac{F_1^*}{\frac{8}{3}k_4} = \bar{\mathcal{H}}_{0,4}^{(1)}$ (since $\epsilon_4 = \frac{8}{3}k_4$) and (51).

For $n = 6$, the analytical formula (48) gives:

$$\begin{aligned}\bar{\mathcal{H}}_{0,6}^{(1)} = \frac{n^{*2}}{64\eta^{11}a^4}\Bigg\{&\left(-\frac{1575}{32}s^4 + \frac{3465}{64}s^6\right)e^4\cos(4g)\\ &+\Bigg[\left(-\frac{525}{2}s^2 + \frac{1575}{2}s^4 - \frac{17325}{32}s^6\right)e^4\\ &+\left(-525s^2 + 1575s^4 - \frac{17325}{16}s^6\right)e^2\Bigg]\cos(2g)\\ &+\left(-\frac{75}{2} + \frac{1575}{4}s^2 + \frac{17325}{32}s^6 - \frac{14175}{16}s^4\right)e^4\\ &+\left(-100 + 1050s^2 - \frac{4725}{2}s^4 + \frac{5775}{4}s^6\right)e^2\\ &-20 + 210s^2 - \frac{945}{2}s^4 + \frac{1155}{4}s^6\Bigg\}\end{aligned} \quad (54)$$

which is the same result as the first order averaged Hamiltonian $\bar{\mathcal{H}}_{0,6}^{(1)}$ for $J_6$ given by the *MM* program (see Table I).

For higher values of $n$, since the expressions (14) and (15) are exponential in $n$, numerical precision has to be considered with the *MM* program, which uses double precision reals for the coefficients (which means a numerical accuracy of $10^{-16}$). Some coefficients of $\bar{\mathcal{H}}_{0,6}^{(1)}$ have already 12 significant digits (see Table I). For $\bar{\mathcal{H}}_{0,12}^{(1)}$, the smallest coefficient in absolute value is $10669659/65536 = 0.1628060760498046875 \times 10^3$ which has 19 significant digits (note that $65536 = 2^{16}$), while the biggest is $50414138775/128 = 0.3938604591796875 \times 10^9$ which has 16 significant digits.

That is the reason why the Legendre polynomial has been implemented with care, i.e. using the robust recurrence on $l$:

$$(l-m)P_l^m(x) = x(2l-1)P_{l-1}^m(x) - (l+m-1)P_{l-2}^m(x) \quad (55)$$

It is useful because there is a closed-form expression for the starting value:

$$P_m^m(x) = (-1)^m(2m-1)!!\,\left(1-x^2\right)^{m/2} \quad (56)$$

Using (55) with $l = m+1$, and setting $P_{m-1}^m(x) = 0$, we find

$$P_{m+1}^m(x) = x(2m+1)P_m^m(x) \quad (57)$$

Equations (56) and (57) provide the two starting values required for (55) for general $l$.

Let's now consider a few checks of (49) for $n$ odd.

For $n = 3$, the analytical formula (49) gives $\bar{\mathcal{H}}_{0,3}^{(1)} = \frac{n^{*2}}{8\eta^5 a}(15s^3 - 12s)e\sin(g)$ which is the same result as Brouwer's (Brouwer, 1959, p. 390) first order averaged Hamiltonian for $J_3$:

$$F_1^* = \Delta_3 F_{2p}^* = \frac{\mu^5 A_{3.0}}{L^3 G^5} e \sin i \left( \frac{3}{8} - \frac{15}{8} \frac{H^2}{G^2} \right) \sin(g) \tag{58}$$

if we consider that $\frac{F_1^*}{A_{3.0}} = \bar{\mathcal{H}}_{0,3}^{(1)}$ (since $\epsilon_3 = A_{3.0}$) and (51).

For $n = 5$, the analytical formula (49) gives:

$$\bar{\mathcal{H}}_{0,5}^{(1)} = \frac{n^{*2}}{32\eta^9 a^3} \left\{ \left[ \left( 90s - 315s^3 + \frac{945}{4}s^5 \right) e^3 + \left( 120s - 420s^3 + 315s^5 \right) e \right] \sin(g) + \left( 35s^3 - \frac{315}{8}s^5 \right) e^3 \sin(3g) \right\} \tag{59}$$

which is the same result as Brouwer's (Brouwer, 1959, p. 391) first order averaged Hamiltonian for $J_5$:

$$F_1^* = \Delta_5 F_{2p}^* = \frac{\mu^7 A_{5.0}}{L^3 G^9} e \sin i \left\{ \frac{15}{128} \left( 1 - 14\frac{H^2}{G^2} + 21\frac{H^4}{G^4} \right) \left( 7 - 3\frac{G^2}{L^2} \right) \sin g - \frac{35}{256} \left( 1 - 10\frac{H^2}{G^2} + 9\frac{H^4}{G^4} \right) \left( 1 - \frac{G^2}{L^2} \right) \sin 3g \right\} \tag{60}$$

if we consider that $\frac{F_1^*}{A_{5.0}} = \bar{\mathcal{H}}_{0,5}^{(1)}$ (since $\epsilon_5 = A_{5.0}$) and (51).

For $n = 7$, the analytical formula (49) gives:

$$\begin{aligned} \bar{\mathcal{H}}_{0,7}^{(1)} = \frac{n^{*2}}{128\eta^{13} a^5} \Bigg\{ & \left( -\frac{2079}{32}s^5 + \frac{9009}{128}s^7 \right) e^5 \sin(5g) \\ & + \left[ \left( -\frac{4725}{8}s^3 + \frac{51975}{32}s^5 - \frac{135135}{128}s^7 \right) e^5 \right. \\ & \left. + \left( -1575s^3 + \frac{17325}{4}s^5 - \frac{45045}{16}s^7 \right) e^3 \right] \sin(3g) \\ & + \left[ \left( -525s + \frac{14175}{4}s^3 - \frac{51975}{8}s^5 + \frac{225225}{64}s^7 \right) e^5 \right. \\ & + \left( -2100s + 14175s^3 - \frac{51975}{2}s^5 + \frac{225225}{16}s^7 \right) e^3 \\ & \left. + \left( -840s + 5670s^3 - 10395s^5 + \frac{45045}{8}s^7 \right) e \right] \sin(g) \quad \Bigg\} \end{aligned} \tag{61}$$

which is the same result as the first order averaged Hamiltonian $\bar{\mathcal{H}}_{0,7}^{(1)}$ for $J_7$ given by the $MM$ program (see Table II).

TABLE II The $\bar{\mathcal{H}}_{0,7}^{(1)}$ series for $J_7$ (20 terms)

| | $f$ | $g$ | $\xi$ | $a$ | $n^*$ | $e$ | $\eta$ | $c$ | $s$ | $(f-l)$ | Coefficient |
|---|---|---|---|---|---|---|---|---|---|---|---|
| sin | 0 | 1 | 0 | −5 | 2 | 1 | −13 | 0 | 1 | 0 | $-0.6562500000000000 \times 10^{1}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 1 | −13 | 0 | 3 | 0 | $0.4429687500000000 \times 10^{2}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 1 | −13 | 0 | 5 | 0 | $-0.8121093750000000 \times 10^{2}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 1 | −13 | 0 | 7 | 0 | $0.4398925781250000 \times 10^{2}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 3 | −13 | 0 | 1 | 0 | $-0.1640625000000000 \times 10^{2}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 3 | −13 | 0 | 3 | 0 | $0.1107421875000000 \times 10^{3}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 3 | −13 | 0 | 5 | 0 | $-0.2030273437500000 \times 10^{3}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 3 | −13 | 0 | 7 | 0 | $0.1099731445312500 \times 10^{3}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 5 | −13 | 0 | 1 | 0 | $-0.4101562500000000 \times 10^{1}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 5 | −13 | 0 | 3 | 0 | $0.2768554687500000 \times 10^{2}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 5 | −13 | 0 | 5 | 0 | $-0.5075683593750000 \times 10^{2}$ |
| sin | 0 | 1 | 0 | −5 | 2 | 5 | −13 | 0 | 7 | 0 | $0.2749328613281250 \times 10^{2}$ |
| sin | 0 | 3 | 0 | −5 | 2 | 3 | −13 | 0 | 3 | 0 | $-0.1230468750000000 \times 10^{2}$ |
| sin | 0 | 3 | 0 | −5 | 2 | 3 | −13 | 0 | 5 | 0 | $0.3383789062500000 \times 10^{2}$ |
| sin | 0 | 3 | 0 | −5 | 2 | 3 | −13 | 0 | 7 | 0 | $-0.2199462890625000 \times 10^{2}$ |
| sin | 0 | 3 | 0 | −5 | 2 | 5 | −13 | 0 | 3 | 0 | $-0.4614257812500000 \times 10^{1}$ |
| sin | 0 | 3 | 0 | −5 | 2 | 5 | −13 | 0 | 5 | 0 | $0.1268920898437500 \times 10^{2}$ |
| sin | 0 | 3 | 0 | −5 | 2 | 5 | −13 | 0 | 7 | 0 | $-0.8247985839843750 \times 10^{1}$ |
| sin | 0 | 5 | 0 | −5 | 2 | 5 | −13 | 0 | 5 | 0 | $-0.5075683593750000 \times 10^{0}$ |
| sin | 0 | 5 | 0 | −5 | 2 | 5 | −13 | 0 | 7 | 0 | $0.5498657226562500 \times 10^{0}$ |

TABLE III Number of terms before and after the integration; efficiency of the analytical result

| $n$ | $\alpha = \# \left\{ \frac{\mathcal{H}^{(0)}_{1,n}}{\xi^2 \eta} \right\}$ | $\beta = \# \left\{ \bar{\mathcal{H}}^{(1)}_{0,n} \right\}$ | $\left\lfloor \frac{\alpha}{\beta} \right\rfloor$ | $\left\lfloor \frac{3n}{2} \right\rfloor$ |
|---|---|---|---|---|
| 2 | 7 | 2 | 3 | 3 |
| 3 | 18 | 2 | 9 | 4 |
| 4 | 48 | 8 | 6 | 6 |
| 5 | 90 | 8 | 11 | 7 |
| 6 | 174 | 20 | 8 | 9 |
| 7 | 280 | 20 | 14 | 10 |
| 8 | 460 | 40 | 11 | 12 |
| 9 | 675 | 40 | 16 | 13 |
| 10 | 1005 | 70 | 14 | 15 |
| 11 | 1386 | 70 | 19 | 16 |
| 12 | 1932 | 112 | 17 | 18 |
| 13 | 2548 | 112 | 22 | 19 |
| 70 | 1610910 | 15540 | 103 | 105 |
| 71 | 1702296 | 15540 | 109 | 106 |

## 7. Efficiency of the Analytical Expression Giving $\bar{\mathcal{H}}^{(1)}_{0,n}$

Now that the formulae (48–50) have been checked, the question of their efficiency arises. What we did in substance is to delete from (14) all the terms periodic in $f$, which gives more compact results. The stage of averaging the Hamiltonian may now be skipped by using directly (48–50).

To have an idea of the gain regarding the computational aspect, one can compare the number of terms generated before and after the integration, for a given $n$. This is done in Table III, which gives:

1. the number of terms before the integration, that is to say contained in $\Delta$ for $n$ even and in $\Delta^*$ for $n$ odd, and after having fully expanded the expression (no powers in the trigonometric functions),
2. the number of terms after the integration, that is to say given by formulae (48–50),
3. the efficiency of the analytical result, i.e. the ratio of the 2 first numbers,
4. the asymptotic trend for the efficiency.

As said previously, the exact analytical determination of the number of terms may be difficult since some combination of indices do not generate any term. However, one knows by looking at (22) or (36) that the number of terms before the integration must be a polynomial of order 4 in $n$, and of order 3 after the integration while looking at (48) and (49). These polynomials have then been determined by polynomial fitting of the number of terms given by the *MM* program for the cases $n = 2-13$ (separately for odd and even cases).

The number of terms before the integration is given by

$$\alpha = \# \left\{ \frac{\mathcal{H}_{1,n}^{(0)}}{\xi^2 \eta} \right\} = \begin{cases} \frac{(8+10n+5n^2+n^3)n}{16} & \text{for } n \text{ even,} \\ \frac{(3+7n+5n^2+n^3)n}{16} & \text{for } n \text{ odd.} \end{cases} \tag{62}$$

while the number of terms after the integration is given by

$$\beta = \# \left\{ \bar{\mathcal{H}}_{0,n}^{(1)} \right\} = \begin{cases} \frac{(8+6n+n^2)n}{24} & \text{for } n \text{ even,} \\ \frac{(-3-n+3n^2+n^3)}{24} & \text{for } n \text{ odd.} \end{cases} \tag{63}$$

so that we conclude that the asymptotic efficiency of the analytical formulae (48–50) is $(n^4/16)/(n^3/24) = 3n/2$ for a given $n$ (whatever the parity). This gives already a factor of order 10 for $J_8$, and of order 100 for $J_{70}$.

## 8. Analytical Expression of $\mathcal{W}_{1,n}$ for the $n$ Even Case

By the relation (8) and the definition (12) of $\Delta$, we have:

$$n^* \mathcal{W}_{1,n} = \int_0^l \left( \mathcal{H}_{1,n}^{(0)} - \bar{\mathcal{H}}_{0,n}^{(1)} \right) \mathrm{d}l = \bar{\mathcal{H}}_{0,n}^{(1)} \, (f - l) + \int_{pp} \Delta \, \mathrm{d}f \tag{64}$$

where $\bar{\mathcal{H}}_{0,n}^{(1)}$ is given by (48–50) and where we use the notation $\int_{pp}$ for the integration of the *periodic part* only.

For the $n$ even case, we have split $\Delta$ by the relation (24) into $A$ and $B$ defined at (25), so that we also have the splitting of the generator:

$$n^* \mathcal{W}_{1,n} = \bar{\mathcal{H}}_{0,n}^{(1)} \, (f - l) + n^* \mathcal{W}_{1,n}^A + n^* \mathcal{W}_{1,n}^B \tag{65}$$

along with the definitions

$$n^* \mathcal{W}_{1,n}^A = \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \sum_{i=0}^{n-1} \sum_{p=0}^{\frac{n}{2}-k-1} \alpha_{i,n}(e) \, \gamma_{p,k,n} \int_{pp} \cos^i f \cos(n - 2k - 2p)(f + g) \, \mathrm{d}f$$

$$n^* \mathcal{W}_{1,n}^B = \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \binom{n - 2k}{\frac{n}{2} - k} \sum_{i=0}^{n-1} \alpha_{i,n}(e) \int_{pp} \cos^i f \, \mathrm{d}f \tag{66}$$

Let's first consider $n^* \mathcal{W}_{1,n}^B$. By making use of the relation (A.8) of the Proposition 1 we obtain directly:

$$\int_{pp} \cos^i(f) \, \mathrm{d}f = \frac{1}{2^i} \left\{ \sum_{j=0}^{\lfloor \frac{i-1}{2} \rfloor} 2 \binom{i}{j} \frac{\sin(i - 2j)f}{i - 2j} \right\} \equiv \frac{C}{2^i} \tag{67}$$

for any $i$ (odd or even).

Let's now consider $n^* \mathcal{W}_{1,n}^A$. We can again separate $f$ from $g$ in the integrand by using the formula $\cos(u + v) = \cos u \cos v - \sin u \sin v$:

$$\int_{pp} \cos^i f \cos(n-2k-2p)(f+g)\ \mathrm{d}f$$
$$= \cos(n-2k-2p)g \int_{pp} \cos^i f \cos(n-2k-2p)f\ \mathrm{d}f$$
$$- \sin(n-2k-2p)g \int_{pp} \cos^i f \sin(n-2k-2p)f\ \mathrm{d}f \tag{68}$$

We apply now the Proposition 3 of the Appendix C with $a = (n-2k-2p)$:

$$\int_{pp} \cos^i(f)\ \sin((n-2k-2p)f)\ \mathrm{d}f$$
$$= \frac{1}{2^i}\left\{ \sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor} \binom{i}{j} \left\{ \frac{-\cos(i-2j+n-2k-2p)f}{i-2j+n-2k-2p} \right\} \right.$$
$$+ \sum_{\substack{j=0\\ j\neq j^*}}^{\lfloor\frac{i-1}{2}\rfloor} \binom{i}{j} \left\{ \frac{\cos(i-2j-(n-2k-2p))f}{i-2j-(n-2k-2p)} \right\}$$
$$\left. -\lambda_i \binom{i}{\frac{i}{2}} \frac{\cos((n-2k-2p)f)}{n-2k-2p} \right\} \equiv \frac{CS}{2^i} \tag{69}$$

and the Proposition 2 of the Appendix B with $a = (n-2k-2p)$:

$$\int_{pp} \cos^i(f) \cos((n-2k-2p)f)\ \mathrm{d}f$$
$$= \frac{1}{2^i}\left\{ \sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor} \binom{i}{j} \left\{ \frac{\sin(i-2j+n-2k-2p)f}{i-2j+n-2k-2p} \right\} \right.$$
$$+ \sum_{\substack{j=0\\ j\neq j^*}}^{\lfloor\frac{i-1}{2}\rfloor} \binom{i}{j} \left\{ \frac{\sin(i-2j-(n-2k-2p))f}{i-2j-(n-2k-2p)} \right\}$$
$$\left. +\lambda_i \binom{i}{\frac{i}{2}} \frac{\sin((n-2k-2p)f)}{n-2k-2p} \right\} \equiv \frac{CC}{2^i} \tag{70}$$

by redefining $j^* = \frac{i-a}{2} = \frac{i-(n-2k-2p)}{2}$. We end up to:

$$2^i \int_{pp} \cos^i f \cos(n-2k-2p)(f+g)\,\mathrm{d}f$$
$$= CC\ \cos(n-2k-2p)g - CS \sin(n-2k-2p)g$$

## 9. Analytical Expression of $\mathcal{W}_{1,n}$ for the $n$ Odd Case

One can do the same kind of developments assuming $n$ odd now (and $n \geq 3$). We work then with $\Delta^*$ defined at (39) and consider only the integration of the periodic part of $A^*$ (there is no $B^*$ for the $n$ odd case) defined at (40):

$$n^*\mathcal{W}_{1,n}^{A,*} = \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \sum_{i=0}^{n-1} \sum_{p=0}^{\frac{n}{2}-k-1} \alpha_{i,n}(e) \gamma^*_{p,k,n} \times \int_{pp} \cos^i f \sin(n-2k-2p)(f+g)\,\mathrm{d}f$$

We can again separate $f$ from $g$ in the integrand by using the formula $\sin(u+v) = \sin u \cos v + \cos u \sin v$ and apply the Proposition 3 of the Appendix C with $a = (n-2k-2p)$, together with the Proposition 2 of the Appendix B; we end up this time to

$$2^i \int_{pp} \cos^i f \sin(n-2k-2p)(f+g)\,\mathrm{d}f = CS\cos(n-2k-2p)g + CC\sin(n-2k-2p)g$$

along with the same definitions (69) and (70) for $CS$ and $CC$.

## 10. Summary of the Analytical Expression Giving $\mathcal{W}_{1,n}$ for any $n$

In summary, we have obtained an analytical formula which gives the first order generator for any zonal harmonic.

For the case $n$ even, we have:

$$n^*\mathcal{W}_{1,n} = \bar{\mathcal{H}}_{0,n}^{(1)}(f-l) + \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \sum_{i=0}^{n-1} \frac{\alpha_{i,n}(e)}{2^i} \left[ C\binom{n-2k}{\frac{n}{2}-k} + \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor - k} \gamma^*_{p,k,n} \{CC\cos(n-2k-2p)g - CS\sin(n-2k-2p)g\} \right] \quad (72)$$

while for the case $n$ odd, we have:

$$n^*\mathcal{W}_{1,n} = \bar{\mathcal{H}}_{0,n}^{(1)}(f-l) + \delta_n \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \beta_{k,n}(s) \sum_{i=0}^{n-1} \frac{\alpha_{i,n}(e)}{2^i} \times \left[ \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor - k} \gamma^*_{p,k,n} \{CS\cos(n-2k-2p)g + CC\sin(n-2k-2p)g\} \right] \quad (73)$$

where $\bar{\mathcal{H}}_{0,n}^{(1)}$ is given by (48–50) and with the definitions (50) completed by:

$$C=\sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor}2\binom{i}{j}\frac{\sin(i-2j)f}{i-2j} \tag{74}$$

$$CS=\sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{j}\left\{\frac{-\cos(i-2j+n-2k-2p)f}{i-2j+n-2k-2p}\right\}$$
$$+\sum_{\substack{j=0\\ j\neq j^*}}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{j}\left\{\frac{\cos(i-2j-(n-2k-2p))f}{i-2j-(n-2k-2p)}\right\}-\lambda_i\binom{i}{\frac{i}{2}}\frac{\cos((n-2k-2p)f)}{n-2k-2p} \tag{75}$$

$$CC=\sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{j}\left\{\frac{\sin(i-2j+n-2k-2p)f}{i-2j+n-2k-2p}\right\}$$
$$+\sum_{\substack{j=0\\ j\neq j^*}}^{\lfloor\frac{i-1}{2}\rfloor}\binom{i}{j}\left\{\frac{\sin(i-2j-(n-2k-2p))f}{i-2j-(n-2k-2p)}\right\}+\lambda_i\binom{i}{\frac{i}{2}}\frac{\sin((n-2k-2p)f)}{n-2k-2p} \tag{76}$$

and with $j^*=\frac{i-(n-2k-2p)}{2}$ we have a similar notable symmetry like before between the odd and even case. Note also that the generator $\mathcal{W}_{1,n}$ may be shifted by a constant (independent of $f$) as usual.

## 11. Checks of the Analytical Expression of $\mathcal{W}_{1,n}$ for $J_2$ to $J_8$

Many checks of our analytical results (72–76) were again performed, mainly by comparison with the results from a symbolic manipulation software (Maple). We would find it cumbersome to give the details here. The formulae have been validated for $J_2$ to $J_8$.

## 12. Conclusions

We have established an analytical formula which gives directly the first order averaged Hamiltonian $\bar{\mathcal{H}}_{0,n}^{(1)}$ for any zonal effect. The analytical formula is efficient since it allows us to skip the averaging process, which means an asymptotic gain of a factor $3n/2$ regarding the computational cost of the $n^{\text{th}}$ zonal. The effects of the zonals could then be easily added linearly to higher order theories. Moreover, the analytical formulae give more physical insight by the fact that we can access more easily a given order of $g$.

We used three symbolic manipulation software, among which the *MM*. We made extensive checks, by comparison either with the results of the *MM* or with published works (Brouwer, 1959). It gives good confidence in our symbolic procedure, hence in new results obtained with the same tool, but for other kind of perturbations, which eventually includes all angles $l, g, h$. As usual for such developments, extreme caution has been taken in order to avoid any error, even typographical.

One could easily deduce the first order averaged equations of motion from this analytical result.

We gave also a similar explicit expression of the generator $\mathcal{W}_{1,n}$ in closed form, which could be used as a first step to obtain, in the future, a second-order theory, or to calculate the explicit expressions of the direct and inverse change of variables in the current first-order theory.


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## Appendix

### A. Proposition giving $\int \cos^i(x)\,\mathrm{d}x$ and $\int_0^{2\pi} \cos^i(x)\,\mathrm{d}x$

PROPOSITION 1. If we define $\lambda_i = 1 - \lceil \frac{i}{2} \rceil + \lfloor \frac{i}{2} \rfloor = \begin{cases} 1 & \text{for } i \text{ even,} \\ 0 & \text{for } i \text{ odd,} \end{cases}$ then we have, for any $i$ (odd or even):

$$\int \cos^i(x)\,\mathrm{d}x = \frac{1}{2^i}\left\{\sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor} 2\binom{i}{j}\frac{\sin(i-2j)x}{(i-2j)} + \lambda_i\binom{i}{\frac{i}{2}}x\right\}$$

$$\int_0^{2\pi} \cos^i(x)\,\mathrm{d}x = \lambda_i \frac{1}{2^i}\binom{i}{\frac{i}{2}}2\pi$$

*Proof*. The function $\cos^i x$ may be expanded (see **1.320** 1. of (Gradshteyn and Ryzhik, 1980)) in that way:

$$\cos^{2n}(x) = \frac{1}{2^{2n}}\left\{\sum_{k=0}^{n-1} 2\binom{2n}{k}\cos 2(n-k)x + \binom{2n}{n}\right\} \tag{A.1}$$

$$\cos^{2n-1}(x) = \frac{1}{2^{2n-2}}\sum_{k=0}^{n-1}\binom{2n-1}{k}\cos(2n-2k-1)x \tag{A.2}$$

which can be rewritten as

$$\cos^i(x) = \frac{1}{2^i}\left\{\sum_{j=0}^{\frac{i}{2}-1} 2\binom{i}{j}\cos(i-2j)x + \binom{i}{\frac{i}{2}}\right\} \text{ for i even,}$$

$$\frac{1}{2^{i-1}}\sum_{j=0}^{\frac{i-1}{2}}\binom{i}{j}\cos(i-2j)x \text{ for } i \text{ odd.} \tag{A.3}$$

and compacted into a formula valid for any $i$ (odd or even):

$$\cos^i(x) = \frac{1}{2^i}\left\{\sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor} 2\binom{i}{j}\cos(i-2j)x + \lambda_i\binom{i}{\frac{i}{2}}\right\} \tag{A.4}$$

by defining

$$\lambda_i = 1 - \lceil\frac{i}{2}\rceil + \lfloor\frac{i}{2}\rfloor = \begin{cases} 1 & \text{for } i \text{ even,} \\ 0 & \text{for } i \text{ odd.}\end{cases} \tag{A.5}$$

The integration of (A.4) gives – see also **2.513** 3. and 4. of (Gradshteyn and Ryzhik, 1980):

$$\int \cos^i(x)\ \mathrm{d}x = \frac{1}{2^i}\left\{\sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor} 2\binom{i}{j}\frac{\sin(i-2j)x}{(i-2j)} + \lambda_i\binom{i}{\frac{i}{2}}x\right\} \tag{A.6}$$

note by the way that $(i-2j)$ never reaches zero for any $j$ in the range.

Hence we have in particular for the mean value:

$$\int_0^{2\pi} \cos^i(x)\ \mathrm{d}x = \lambda_i\frac{1}{2^i}\binom{i}{\frac{i}{2}}2\pi \tag{A.7}$$

while for the integration of the *periodic part* only (what we will remind by the symbol $\int_{pp}$) of $\cos^i(x)$, we have :

$$\int_{pp} \cos^i(x)\ \mathrm{d}x = \frac{1}{2^i}\left\{\sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor} 2\binom{i}{j}\frac{\sin(i-2j)x}{i-2j}\right\} \text{for any } i \text{ (odd or even).} \tag{A.8}$$

■

These formulae have been checked by the use of a symbolic manipulation software (Mathematica). For example, the integration done by Mathematica gives, for $n \in \mathbb{N}$:

$$\int_0^{2\pi} \cos^{2n+1}(x)\ \mathrm{d}x = 0 \tag{A.9}$$

which checks directly the odd case. For the $i$ even case, we had the result:

$$\int_0^{2\pi} \cos^i(x)\ \mathrm{d}x = \frac{1}{2^i}\binom{i}{\frac{i}{2}}2\pi \tag{A.10}$$

which can be rewritten

$$\int_0^{2\pi} \cos^{2n}(x)\ \mathrm{d}x = \frac{1}{2^{2n}}\binom{2n}{n}2\pi \tag{A.11}$$

Now Mathematica gives:

$$\int_0^{2\pi} \cos^{2n}(x)\ \mathrm{d}x = \frac{2^{1+2n}\pi^2}{(2n)!\left(\Gamma\left(\frac{1}{2}-n\right)\right)^2} \tag{A.12}$$

a result which refers to the gamma function (Euler's integral of the second kind), reduced to the case $n \in \mathbb{N}$. So that we have to expand also the r.h.s. of (A.11) into gamma functions:

$$\frac{1}{2^{2n}}\binom{2n}{n}2\pi = 2\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma(1+n)} \tag{A.13}$$

Finally the following equation remains to be checked:

$$\frac{2^{1+2n}\pi^2}{(2n)!\left(\Gamma\left(\frac{1}{2}-n\right)\right)^2} = 2\sqrt{\pi}\frac{\Gamma\left(\frac{1}{2}+n\right)}{\Gamma(1+n)} \tag{A.14}$$

which turns out to be true if we consider the following relationships, valid for $n \in \mathbb{N}$ (see **8.339** of (Gradshteyn and Ryzhik, 1980)):

$$\begin{aligned}
\Gamma(n) &= (n-1)! \\
\Gamma\left(n+\frac{1}{2}\right) &= \frac{\sqrt{\pi}}{2^n}(2n-1)!! \\
\Gamma\left(\frac{1}{2}-n\right) &= (-1)^n\frac{2^n\sqrt{\pi}}{(2n-1)!!} \\
(2n)! &= (2)^n n!(2n-1)!!
\end{aligned} \tag{A.15}$$

where $(2n-1)!!$ means the product of all odd integers less or equal to $(2n-1)$.

## B. Proposition giving $\int_{pp} \cos^i(f)\cos(af)\,\mathrm{d}f$ and $\int_0^{2\pi} \cos^i(f)\ \cos(af)\ \mathrm{d}f$

PROPOSITION 2. For $(i, a) \in \mathbb{N}$, and $a > 0$, we have

$$\int_0^{2\pi} \cos^i(f)\cos(af)\ \mathrm{d}f = \begin{cases} \dfrac{\pi}{2^{i-1}}\dbinom{i}{\frac{i-a}{2}} & \text{if } (i,\ a) \text{ have the same parity, and } (a \leqslant i) \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\int_{pp} \cos^i(f)\cos(af)\,\mathrm{d}f$$

$$= \frac{1}{2^i}\left\{ \sum_{j=0}^{\lfloor \frac{i-1}{2} \rfloor} \binom{i}{j} \left\{ \frac{\sin(i-2j+a)f}{i-2j+a} \right\} + \sum_{\substack{j=0\\ j\neq j^*}}^{\lfloor \frac{i-1}{2} \rfloor} \binom{i}{j} \left\{ \frac{\sin(i-2j-a)f}{i-2j-a} \right\} \right.$$

$$\left. + \lambda_i \binom{i}{\frac{i}{2}} \frac{\sin(af)}{a} \right\}$$

for any parity of $i$ and with $j^* = \frac{i-a}{2}$ and with

$$\lambda_i = 1 - \lceil \frac{i}{2} \rceil + \lfloor \frac{i}{2} \rfloor = \begin{cases} 1 & \text{for } i \text{ even,} \\ 0 & \text{for } i \text{ odd.} \end{cases}$$

*Proof (for the case i odd)*. The function $\cos^i f$ may be expanded by (A.3), which gives:

$$\int_0^{2\pi} \cos^i(f)\cos(af)\,\mathrm{d}f$$

$$= \int_0^{2\pi} \frac{1}{2^{i-1}} \sum_{j=0}^{\frac{i-1}{2}} \binom{i}{j} \cos(i-2j)f \cos(af)\,\mathrm{d}f$$

$$= \frac{1}{2^{i-1}} \sum_{j=0}^{\frac{i-1}{2}} \binom{i}{j} \int_0^{2\pi} \frac{1}{2} \{\cos(i-2j+a)f + \cos(i-2j-a)f\}\,\mathrm{d}f \qquad \text{(B.1)}$$

by using the formula $2\cos u \cos v = \cos(u+v) + \cos(u-v)$. But all trigonometric functions have null contribution, except in the case $(*)(i-2j+a) = 0 \leftrightarrow j = \frac{i+a}{2}$ or $(**)$ $(i-2j-a) = 0 \leftrightarrow j = \frac{i-a}{2}$. As $j$ must $\in \mathbb{N}$, $i$ and $a$ must have same parity. Since the upper bound of $j$ is $\frac{i-1}{2}$, it will never reach $\frac{i+a}{2}$ since $a \in \mathbb{N}$. So only the case $(**)$ $j = \frac{i-a}{2} = j^*$ need to be considered; with the final constraint that $i \geqslant a$.

So (B.1) becomes:

$$\int_0^{2\pi} \cos^i(f)\cos(af)\,\mathrm{d}f$$

$$= \begin{cases} \dfrac{\pi}{2^{i-1}} \dbinom{i}{\frac{i-a}{2}} & \text{if } (i,\, a) \text{ have the same parity, and } (a \leqslant i) \\ 0 & \text{otherwise.} \end{cases} \qquad \text{(B.2)}$$

Moreover, if we consider now the integration of the periodic part only, we obtain:

$$
\begin{aligned}
&\int_{pp} \cos^i(f)\cos(af)\ \mathrm{d}f \\
&= \frac{1}{2^{i-1}} \sum_{j=0}^{\frac{i-1}{2}} \binom{i}{j} \int_{pp} \frac{1}{2}\{\cos(i-2j+a)f + \cos(i-2j-a)f\}\ \mathrm{d}f \\
&= \frac{1}{2^i}\left\{ \sum_{j=0}^{\frac{i-1}{2}} \binom{i}{j}\left\{\frac{\sin(i-2j+a)f}{i-2j+a}\right\} + \sum_{\substack{j=0\\ j\neq j^*}}^{\frac{i-1}{2}} \binom{i}{j}\left\{\frac{\sin(i-2j-a)f}{i-2j-a}\right\}\right\}
\end{aligned} \tag{B.3}
$$

■

*Proof (for the case i even)*. The function cos $^if$ may be expanded by (A.3), which gives:

$$
\begin{aligned}
\int_0^{2\pi} \cos^i(f)\cos(af)\,\mathrm{d}f &= \int_0^{2\pi} \frac{1}{2^i}\left\{\sum_{j=0}^{\frac{i}{2}-1} 2\binom{i}{j}\cos(i-2j)f + \binom{i}{\frac{i}{2}}\right\}\cos(af)\,\mathrm{d}f \\
&= \frac{1}{2^i}\left\{\sum_{j=0}^{\frac{i}{2}-1} 2\binom{i}{j}\int_0^{2\pi}\frac{1}{2}\Big[\cos(i-2j+a)f \right. \\
&\left. \quad + \cos(i-2j-a)f\Big]\,\mathrm{d}f + \binom{i}{\frac{i}{2}}\int_0^{2\pi}\cos(af)\,\mathrm{d}f\right\}
\end{aligned} \tag{B.4}
$$

The trigonometric function cos($af$) will have a null contribution since $a \in \mathbb{N}$ and $a > 0$. The other trigonometric functions will also have null contribution, except again in the case $(*)(i-2j+a) = 0 \leftrightarrow j = \frac{i+a}{2}$ or $(**)(i-2j-a) = 0 \leftrightarrow j = \frac{i-a}{2}$. As $j$ must $\in \mathbb{N}$, $i$ and $a$ must have same parity. Since the upper bound of $j$ is $\frac{i-1}{2}$, it will never reach $\frac{i+a}{2}$ since $a \in \mathbb{N}$. So only the case $(**)\ j = \frac{i-a}{2}$ need to be considered; with the final constraint that $i \geqslant a$.

So (B.4) becomes:

$$
\int_0^{2\pi} \cos^i(f)\cos(af)\,\mathrm{d}f = \begin{cases} \frac{\pi}{2^{i-1}}\binom{i}{\frac{i-a}{2}} & \text{if } (i,a) \text{ are of the same parity, and } (a \leqslant i) \\ 0 & \text{otherwise} \end{cases} \tag{B.5}
$$

Moreover, if we consider now the integration of the periodic part only, we obtain:

$$
\begin{aligned}
&\int_{pp} \cos^i(f)\cos(af)\ \mathrm{d}f \\
&= \frac{1}{2^i}\left\{\sum_{j=0}^{\frac{i}{2}-1} 2\binom{i}{j}\int_{pp}\frac{1}{2}\Big\{\cos(i-2j+a)f + \cos(i-2j-a)f\Big\}\ \mathrm{d}f \right. \\
&\left. \quad + \binom{i}{\frac{i}{2}}\int_{pp}\cos(af)\ \mathrm{d}f\right\} = \frac{1}{2^i}\left\{\sum_{j=0}^{\frac{i}{2}-1}\binom{i}{j}\left\{\frac{\sin(i-2j+a)f}{i-2j+a}\right\}\right. \\
&\left. \quad + \sum_{\substack{j=0\\ j\neq j^*}}^{\frac{i}{2}-1}\binom{i}{j}\left\{\frac{\sin(i-2j-a)f}{i-2j-a}\right\} + \binom{i}{\frac{i}{2}}\frac{\sin(af)}{a}\right\}
\end{aligned} \tag{B .6}
$$

■

Since (B.2) and (B.5) are equivalent, the Proposition 2 is valid for any parity of $i$ concerning the constant part. Concerning the periodic part, (B.3) and (B.6) may be gathered into:

$$\int_{pp} \cos^i(f)\cos(af)\,\mathrm{d}f$$
$$= \frac{1}{2^i}\left\{ \sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor} \binom{i}{j} \left\{\frac{\sin(i-2j+a)f}{i-2j+a}\right\} + \sum_{\substack{j=0\\ j\neq j^*}}^{\lfloor\frac{i-1}{2}\rfloor} \binom{i}{j} \left\{\frac{\sin(i-2j-a)f}{i-2j-a}\right\}\right.$$
$$\left. +\lambda_i \binom{i}{\frac{i}{2}} \frac{\sin(af)}{a} \right\}$$

which is now valid for any parity of $i$ by using (A.5), so that the Proposition 2 is also valid for any parity of $i$ concerning the periodic part.

*Alternate proof of* Proposition 2 *concerning the constant part*. One could also consider the following relationship (see **3.631** 9. of (Gradshteyn and Ryzhik, 1980)), valid for Re $v > 0$:

$$\int_0^{\frac{\pi}{2}} \cos^{v-1}(x)\cos(ax)\,\mathrm{d}x = \frac{\pi}{2^v v B\left(\frac{v+a+1}{2}, \frac{v-a+1}{2}\right)} \tag{B.8}$$

a result which refers to the beta function (Euler's integral of the first kind), which, in the case $(n, m) \in \mathbb{N}^2$, reduces to (see **8.384** 6. of (Gradshteyn and Ryzhik, 1980)):

$$\frac{1}{B(n,m)} = m\binom{n+m-1}{n-1} = n\binom{n+m-1}{m-1} \tag{B.9}$$

so that (B.8) becomes:

$$\int_0^{\frac{\pi}{2}} \cos^{v-1}(x)\cos(ax)\,\mathrm{d}x = \frac{\pi}{2^v v}\frac{v+a+1}{2}\binom{v}{\frac{v-a-1}{2}} \tag{B.10}$$

If we now substitute $i$ for $v-1$ in order to finally compare with Proposition 2, and if we use the relationship $\binom{i+1}{n} = \frac{i+1}{i+1-n}\binom{i}{n}$ one obtains successively:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^{v-1}(x)\cos(ax)\,\mathrm{d}x &= \int_0^{\frac{\pi}{2}} \cos^{i}(x)\cos(ax)\,\mathrm{d}x \\
&= \frac{\pi}{2^{i+1}(i+1)}\frac{i+2+a}{2}\binom{i+1}{\frac{i-a}{2}} \\
&= \frac{\pi}{2^{i+1}}\frac{i+2+a}{2}\frac{1}{i+1-\frac{i-a}{2}}\binom{i}{\frac{i-a}{2}} \\
&= \frac{\pi}{2^i+1}\binom{i}{\frac{i-a}{2}}
\end{aligned} \tag{B.11}$$

And finally, from symmetry considerations, we have:

$$\int_0^{2\pi} \cos^i(x)\cos(ax)\,\mathrm{d}x$$
$$= \begin{cases} 4\displaystyle\int_0^{\frac{\pi}{2}} \cos^i(x)\cos(ax)\,\mathrm{d}x & \text{if } (i,a) \text{ are of the same parity, and } (a \leqslant i) \\ 0 & \text{otherwise.} \end{cases}$$
$$= \begin{cases} \dfrac{\pi}{2^{i-1}}\dbinom{i}{\frac{i-a}{2}} & \text{if } (i,a) \text{ are of the same parity, and } (a \leqslant i) \\ 0 & \text{otherwise.} \end{cases} \tag{B.12}$$

which proofs Proposition 2 again concerning the constant part. ■

## C. Proposition giving $\int_{pp} \cos^i(f)\sin(af)\,\mathrm{d}f$ and $\int_0^{2\pi} \cos^i(f)\sin(af)\,\mathrm{d}f$

PROPOSITION 3. If we define $j^* = \frac{i-a}{2}$ and

$$\lambda_i = 1 - \lceil\frac{i}{2}\rceil + \lfloor\frac{i}{2}\rfloor = \begin{cases} 1 & \text{for } i \text{ even,} \\ 0 & \text{for } i \text{ odd} \end{cases}$$

then we have, for any $i$ (odd or even):

$$\int_0^{2\pi} \cos^i(f)\sin(af)\,\mathrm{d}f = 0 \text{ for } (i,a) \in \mathbb{N},$$

and

$$\int_{pp} \cos^i(f)\sin(af)\,\mathrm{d}f$$
$$= \frac{1}{2^i}\left\{ \sum_{j=0}^{\lfloor\frac{i-1}{2}\rfloor} \binom{i}{j} \left\{ \frac{-\cos(i-2j+a)f}{i-2j+a} \right\} \right.$$
$$\left. + \sum_{\substack{j=0 \\ j\neq j^*}}^{\lfloor\frac{i-1}{2}\rfloor} \binom{i}{j} - \left\{ \frac{\cos(i-2j-a)f}{i-2j-a} \right\} - \lambda_i \binom{i}{\frac{i}{2}} \frac{\cos(af)}{a} \right\}$$

*Proof*. The proof is very similar to the proof of the Proposition 2. The function $\cos^i f$ may be expanded as before by (A.3). Then we use now the formula $2\cos u \sin v = \sin(u+v) - \sin(u-v)$, so that we are left with only sin trigonometric functions. These have null contribution when being integrated over $2\pi$, while they give cos functions when being integrated for the periodic part. ■

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